NORMS ON QUOTIENT SPACES

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1. Perturbation classes. Let $\mathcal{S}$ be a subset of a Banach space $\mathfrak{A}$ over the complex numbers, and assume that $\alpha\mathcal{S}\subset\mathcal{S}$ for each scalar $\alpha\neq 0$. Let $P(\mathcal{S})$ denote the set of elements of $\mathfrak{A}$ that perturb S into itself, i.e., $P(\mathcal{S})=\{a\in\mathfrak{A}: a+s\in\mathcal{S} \text{ for all } s\in\mathcal{S}\}$.

PROPOSITION 1.1. *$P(\mathcal{S})$ is a linear subspace of $\mathfrak{A}$. If $\mathcal{S}$ is an open subset of $\mathfrak{A}$, then $P(\mathcal{S})$ is closed.*

PROPOSITION 1.2. *Let $\mathcal{S}_1\subset\mathcal{S}_2$ be two such subsets, and assume that $\mathcal{S}_1$ is open and $\mathcal{S}_2$ does not contain any boundary point of $\mathcal{S}_1$. Then $P(\mathcal{S}_2)\subset P(\mathcal{S}_1)$.*

PROPOSITION 1.3. *Assume that $\mathfrak{A}$ is a Banach algebra with identity e. Let G denote the set of invertible elements in $\mathfrak{A}$. If $G\mathcal{S}\subset\mathcal{S}$, then $P(\mathcal{S})$ is a left ideal. If $\mathcal{S}G\subset\mathcal{S}$, then $P(\mathcal{S})$ is a right ideal.*

PROPOSITION 1.4. *$P(G)=R$, the radical of $\mathfrak{A}$.*

Let G_l (G_r) denote the set of left (right) invertible elements of $\mathfrak{A}$, and let H_l (H_r) denote the set of elements of $\mathfrak{A}$ that are not left (right) topological divisors of zero.

THEOREM 1.5. $P(H_l)\subset P(G_l)=R=P(G_r)\supset P(H_r)$.

Let X be a Banach space, and let $B(X)$ $[\mathcal{K}(X)]$ denote the set of bounded (compact) linear operators on X. Take $\mathfrak{A}=B(X)/\mathcal{K}(X)$ and let π be the canonical homomorphism from $B(X)$ to $\mathfrak{A}$. Set

$$\Phi(X) = \pi^{-1}(G), \qquad \Phi_l(X) = \pi^{-1}(G_l), \qquad \Phi_r(X) = \pi^{-1}(G_r).$$

It is well known [6] that $\Phi_l(X)$ consists of those operators having finite nullity and closed, complemented ranges, and that $\Phi_r(X)$ consists of those operators having complemented null spaces and closed ranges with finite codimensions. $\Phi(X)=\Phi_l(X)\cap\Phi_r(X)$ is the set of Fredholm operators on X.

THEOREM 1.6. $P(\Phi)=P(\Phi_l)=P(\Phi_r)=\pi^{-1}(R)$.

Let Z be any subset of $\{0,\pm 1,\pm 2,\cdots,\pm\infty\}$, and let Φ_z be the collection of those operators $A\in\Phi_l(X)\cup\Phi_r(X)$ such that $i(A)\in Z$, where $i(A)=\dim N(A)-\dim N(A')$.

THEOREM 1.7. $P(\Phi_z)=\pi^{-1}(R)$.