COMPARISON THEOREMS FOR A GENERALIZED MODULUS OF CONTINUITY

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1. **Notation and definitions.** Let R^m denote real Euclidean m-space. We shall employ standard vector notations, whereby $t=(t_1, \cdots, t_m)$, $u=(u_1, \cdots, u_m)$ denote points of R^m, $tu=\sum_1^m t_iu_i$ and $|t|=(tt)^{1/2}$. In connection with Fourier transforms $x=(x_1, \cdots, x_m)$ denotes a point of a "dual" copy of R^m. $M=M(R^m)$ shall denote the totality of bounded complex-valued Borel measures on R^m, made into a Banach algebra in the usual way, i.e. the "product" of the measures σ, τ is the convolution $\sigma * \tau$ and the norm of a measure is its total variation. $\hat{\sigma}$ shall denote the Fourier transform of σ, and $W=W(\hat{R}^m)$ the Banach algebra of Fourier transforms of elements of M. In W the "multiplication" is ordinary point-wise multiplication of functions. W is isometrically isomorphic to M under the map $\sigma \rightarrow \hat{\sigma}$.

For $f \in L^p(R^m)$ with $1 \leqq p \leqq \infty$ and $\sigma \in M$ we write $f * \sigma$ to denote the function g such that $g(t)=\int f(t-u)d\sigma(u)$. (This is defined for almost all t; moreover $g \in L^p$.) For $a>0$ let $\sigma_{(a)}$ denote the measure defined by $\sigma_{(a)}(E)=\sigma(a^{-1}E)$ for all Borel sets E. We have $\|\sigma_{(a)}\|=\|\sigma\|$.

For $f \in L^p$, $\sigma \in M$ and $a>0$ let us define

$$D_{\sigma,p}(f; a) = \|f * \sigma_{(a)}\|_p,$$
$$\omega_{\sigma,p}(f; a) = \sup_{0<b\leqq a} D_{\sigma,p}(f; b).$$

We shall call $\omega_{\sigma,p}$ (for fixed f, considered as a function of a) *the σ, p modulus* of f. The reason for the choice of this term is that when $m=1$ and σ is specialized to be a "dipole measure" with "masses" of $+1$ at $t=0$ and -1 at $t=1$ we obtain the usual L^p modulus of continuity. Various other specializations of σ lead to "moduli" that are of interest in studying the smoothness and approximation properties of functions (see [1], [2]).

We shall say a measure σ *satisfies the Tauberian condition* if $\hat{\sigma}(x)$ does not vanish at all points of any half-ray through the origin, i.e. if given x, $|x|=1$ there is a positive number a such that $\hat{\sigma}(ax) \neq 0$.

2. **The comparison theorems.**

THEOREM 1. *Suppose σ, $\tau \in M(R^m)$, σ satisfies the Tauberian condition, and there is a function $F \in W$ such that $\hat{\tau}(x)=\hat{\sigma}(x)F(x)$ in some neighborhood of the origin. Then for $f \in L^p$ $(1 \leqq p \leqq \infty)$ and $a>0$*