A STURM THEOREM FOR STRONGLY ELLIPTIC SYSTEMS AND APPLICATIONS

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A recent announcement in this Bulletin [1] deals with Sturm comparison theorems for elliptic equations and systems of ordinary differential equations based on a generalization of the Picone identity [2, p. 228]: if $v(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{u}{v}(au'v - guv')\right] = u(au')' - \frac{u^2}{v}(gv')' + (a-g)u'^2 + g\left(u' - \frac{u}{v}v'\right)^2. \tag{1}$$

The purpose of this note is to present a generalization of (1) to strongly elliptic systems and to describe some of its consequences.

Let $A_{ij}(x)$, $G_{ij}(x)$, $C(x)$, and $H(x)$ denote real symmetric $n \times n$ matrices whose components are defined in a smooth, bounded closed domain $\overline{D}$ in R_m and which satisfy $G_{ij} = G_{ji}$, $A_{ij} = A_{ji}$ for $i, j = 1, \cdots, m$. The components of A_{ij} and G_{ij} are to be of class C^2 in $\overline{D}$ while the components of C and H are continuous. Let $V(x)$ be a $n \times n$ matrix of class C^2 which is nonsingular in $\overline{D}$ and is "prepared" in the sense that it satisfies

$$V^* \sum_{j=1}^{m} G_{ij} \frac{\partial V}{\partial x_j} \quad \text{is symmetric for } i = 1, \cdots, m, \tag{2}$$

and let $U(x)$ be a $n \times 1$ matrix of class C^2. Then (1) has the following generalization:

$$\begin{aligned} \sum_i \frac{\partial}{\partial x_i}\left[U^* \sum_j A_{ij} \frac{\partial U}{\partial x_j} - U^* \sum_j G_{ij} \frac{\partial V}{\partial x_j} V^{-1} U\right] &= U^* \sum_{i,j} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial U}{\partial x_i}\right) - U^* \sum_{i,j} \frac{\partial}{\partial x_i}\left(G_{ij} \frac{\partial V}{\partial x_j}\right) V^{-1} U \\ &\quad + \sum_{i,j} \frac{\partial U^*}{\partial x_i}(A_{ij} - G_{ij}) \frac{\partial U}{\partial x_j} \\ &\quad + \sum_{i,j}\left[\frac{\partial U}{\partial x_i} - \frac{\partial V}{\partial x_i} V^{-1} U\right]^* G_{ij}\left[\frac{\partial U}{\partial x_j} - \frac{\partial V}{\partial x_j} V^{-1} U\right]. \end{aligned} \tag{3}$$