STACKED BASES FOR MODULES

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Let R be a principal ideal domain (PID), B a free R-module and A a (necessarily free) submodule. Bases $\{a_j: j \in J\}$ and $\{b_i: i \in I\}$ for A and B, respectively, are said to be *stacked* if $J \subset I$ and for each $j \in J$, there is an element $m_j \in R$ such that $a_j = m_j b_j$. An obvious necessary condition for the existence of stacked bases is that B/A be a direct sum of cyclic modules.

STACKED BASES THEOREM. *Free modules $A \subset B$ over a PID have stacked bases if and only if B/A is a direct sum of cyclic modules.*

The Stacked Bases Theorem was conjectured by Kaplansky in 1954 [**2**, pp. 66, 80][2].

One can interpret this result on the level of chain complexes, as follows. Call a chain complex of R-modules *elementary* if it is isomorphic to one of the form $\cdots 0 \to mR \subset R \to 0 \cdots$, where $m \in R$ and all the unlisted terms are 0. Then we have

COROLLARY. *A chain complex C of modules over a PID is a direct sum of elementary chain complexes if and only if C is free and the homology $H_*(C)$ is a direct sum of cyclic modules.*

The proof is straightforward. Note that applying the corollary to the chain complex $0 \to A \subset B \to 0$ yields the original result.

The Stacked Bases Theorem is well known when B is finitely generated [**3**, p. 162], but the proof in the general case is quite different. In what follows, let us understand all modules to be over a fixed PID. A presentation ξ of a module C is a short exact sequence

$$\xi: 0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

in which B (and therefore also A) are free modules. The presentation is said to be *stacked* if ιA and B have stacked bases. A subscript attached to ξ indicates that the same subscript should be attached to the other characters. Direct sums of presentations are defined in the obvious way. The following lemmas indicate the division of labor in the proof.

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[2] Page references are to the 2nd (1969) edition of Kaplansky's book.