ON RELATIVE GROTHENDIECK RINGS

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Let K be a field of characteristic $p(\neq 0$, to exclude trivial cases) and let G be a finite goup. A KG-module M is a finite dimensional K-vector space, on which G acts K-linearly from the left.

The Green ring $a(G)$ of G(w.r.t. K) is the free abelian group, spanned by the isomorphism classes of indecomposable KG-modules, with the multiplication induced from the tensor product $\oplus_K$ of KG-modules (see [4]).

If $U \leqq G$ one has a restriction map $a(G) \to a(U)$, induced from restricting the action of G on a KG-module M to U, thus getting a KU-module $M|_U$. Let $\mathfrak{U}$ be a family of subgroups of G. An exact sequence

$$E: 0 \to M' \to M \to M'' \to 0$$

is said to be $\mathfrak{U}$-split, if

$$E|_U: 0 \to M'|_U \to M|_U \to M''|_U \to 0$$

is a split exact sequence of KU-modules for any $U \in \mathfrak{U}$.

For any $\mathfrak{U}$-split exact sequence E of KG-modules define $\chi_E = M - M' - M''$ to be its Euler characteristic in $a(G)$. Write $i(G, \mathfrak{U})$ for the linear span of the elements $\chi_E \in a(G)$, where E runs through all $\mathfrak{U}$-split exact sequences of KG-modules. $i(G, \mathfrak{U})$ is an ideal in $a(G)$ and $a(G, \mathfrak{U}) = a(G)/i(G, \mathfrak{U})$ the Grothendieck ring of G relative to $\mathfrak{U}$ (see [1], [6]).

LEMMA 1. *Let $\mathfrak{U}_1$, $\mathfrak{U}_2$ be two families of subgroups of G. Then the multiplication map $a(G) \times a(G) \to a(G)$ sends $i(G, \mathfrak{U}_1) \times i(G, \mathfrak{U}_2)$ into $i(G, \mathfrak{U}_1 \cup \mathfrak{U}_2)$.*

PROOF. If E_i: $0 \to M_i' \to M_i \to M_i'' \to 0$ is exact and $\mathfrak{U}_i$-split, then the tensor product of these two complexes E_1, E_2 is exact and $\mathfrak{U}_1 \cup \mathfrak{U}_2$-split, therefore $\chi_{E_1 \otimes E_2} = \chi_{E_1} \cdot \chi_{E_2} \in i(G, \mathfrak{U}_1 \cup \mathfrak{U}_2)$.

An KG-module M is $\mathfrak{U}$-projective, if M is a direct summand in $\oplus_{U \in \mathfrak{U}} (M|_U)^{U \to G}$ (see [3]), where for a KU-module N we write $N^{U \to G}$ for the induced KG-module $KG \otimes_{KU} N$.

Write $k(G, \mathfrak{U})$ for the linear span of the $\mathfrak{U}$-projective modules in $a(G)$. The canonical epimorphism $a(G) \twoheadrightarrow a(G, \mathfrak{U})$ induces a map κ: $k(G, \mathfrak{U}) \to a(G, \mathfrak{U})$, which has also been called the Cartan map (see [1], [6], [7]).