ON ORBITS ASSOCIATED WITH SYMMETRIC SPACES

BY BERTRAM KOSTANT AND STEPHEN RALLIS

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1. Introduction. The results of [2] on the orbit structure of a complex reductive Lie algebra, under the adjoint group, are generalized to the analogue associated with a symmetric space. Notwithstanding some new complications such as the nonirreducibility of the variety of nilpotent elements and the nonnormality of orbits of maximal dimension the symmetric space case seems to be the correct setting for the results.

The notions of the principal TDS, principal nilpotent element, regular element, the finiteness of the number of nilpotent orbits and the orbit structure as described by the invariant polynomials and nilpotent orbits all go through in the general case. Also (see [3]) the orbits of maximal dimension can be sectioned in an as nice a manner as in the case of the adjoint representation (see [2]).

2. Regular, semisimple and nilpotent elements.

2.1. Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{g}_R$ be a real form fixed throughout. Also let $\mathfrak{g}_R = \mathfrak{k}_R + \mathfrak{p}_R$ be a Cartan decomposition of $\mathfrak{g}_R$ fixed throughout. Thus exp ad $\mathfrak{k}_R$ operating on $\mathfrak{g}_R$ is a maximal compact subgroup of the adjoint group of $\mathfrak{g}_R$ and if

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

is the linear direct sum obtained by complexifying $\mathfrak{k}_R$ and $\mathfrak{p}_R$ then the map θ: $\mathfrak{g} \to \mathfrak{g}$ defined by putting $\theta = 1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$, is a Lie algebra automorphism of order 2.

Now let G be the adjoint group of $\mathfrak{g}$ and let K_θ be the subgroup of elements $a \in G$ which commute with θ. Thus $\mathfrak{k}$ and $\mathfrak{p}$ are stable under the action of K_θ and in particular the latter defines a representation

$$K_\theta \to \operatorname{Aut} \mathfrak{p}$$

which will be of concern to us throughout. One notes of course that the identity component K of K_θ is just the subgroup of G corresponding to $\mathfrak{k}$.

Let $\mathfrak{a}_R$ be a maximal abelian subalgebra of $\mathfrak{p}_R$ and let $\mathfrak{a} \subseteq \mathfrak{p}$ be its complexification. Let $A = \exp \operatorname{ad} \mathfrak{a}$ and let F be the finite group of all elements of order 2 in A. Clearly $F \subseteq K_\theta$ and hence F normalizes K. The relation between K_θ and K is clarified by

PROPOSITION 1. *One has* $K_\theta = FK$.