RULED SURFACES AND THE ALBANESE MAPPING

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1. Much of the classical theory of algebraic curves is summarized by saying there is a map $C(n) \to J$ from the n-fold symmetric product of the curve C into an abelian variety J, the Jacobian, and the fibers are projective spaces (representing the linear systems of degree n). For algebraic surfaces there is an analogous map $V(n) \to A$ from the n-fold symmetric product of the surface V to its Albanese variety. The fibers are irreducible and regular if n is large, but it has been a long open question whether they are rational, or ever can be.

THEOREM. *Let V be a complete nonsingular surface in characteristic zero, and let q denote the dimension of its Albanese variety A. If for some $n > q$ the general fiber of the morphism $V(n) \to A$ is a rational variety, then V is a ruled surface.*

By the "general" fiber we mean as usual that there is an open set in A over which all fibers have the indicated property. If V is ruled, i.e., birationally equivalent to the product $P^1 \times C$ of a projective line and a curve C, then the general fiber is rational for all n: for this converse to the theorem, one needs only the quoted result for curves plus the remark that then the Albanese variety of V is just the Jacobian of C. A proof of the theorem when $q = 0$ was the subject of an earlier paper [2], some of whose ideas recur here. There is also overlap with a recent (independent) proof by Mumford [3] that the rational equivalence ring is not of finite type; both proofs use the idea of bounding the dimension of the zero-locus of a 2-form.

2. **A generic smoothness lemma.** We need the

LEMMA. *Let $f: X \to Y$ be a dominating morphism of varieties in characteristic zero, with X nonsingular and projective. Then f has maximal rank along the general fiber F_y, so F_y is nonsingular.*

PROOF. The lemma is local on Y; by Noether normalization we may reduce to the case where Y is affine r-space, with coordinate functions $x_1, \cdots, x_r$. As a_1 varies over the (algebraically closed) ground field, the zeros of $x_1 - a_1$ on X give a linear system of divisors on X; by Bertini's theorem, a general member—say X_1—is a disjoint union of

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