SEMIGROUPS OF PARTIAL ISOMETRIES

BY LAWRENCE J. WALLEN

Communicated by Paul Halmos, February 24, 1969

Let $\{S_t\}$ be the (strongly continuous) semigroup of operators on $L_2(0, 1)$ defined as follows: if $0 \leqq t < 1$, then

$$\begin{aligned} S_t f(x) &= 0 \qquad \text{if } x \leqq t, \\ &= f(x-t) \qquad \text{if } t < x \leqq 1; \end{aligned}$$

if $t \geqq 1$, then $S_t = 0$. The operators S_t are related to the classical Volterra operator J ($Jf(x) = \int_0^x f(t)dt$) by the equation $J = \int_0^1 S_t dt$ or, what comes to the same thing, $i\lambda J(I - i\lambda J)^{-1} = \int_0^1 e^{i\lambda t} S_t dt$. These formulas, together with the uniqueness of the Fourier transform, permit one to pass readily from considerations concerning J to those about $\{S_t\}$ and vice versa. This correspondence was used by Dixmier in [1] where, however, he considers these operators on L_1.

Let us note three properties of $\{S_t\}$:

(a) each S_t is a partial isometry,

(b) $S_1 = 0$ and $S_t \neq 0$ if $0 \leqq t < 1$,

(c) $\{S_t\}$ is irreducible.

It turns out that (a), (b), and (c) characterize $\{S_t\}$ up to unitary equivalence. This is a special case of the following result.

THEOREM. *If $\{W_t\}$ is a strongly continuous semigroup of operators on a Hilbert space and if $\{W_t\}$ satisfies* (a) *and* (b), *then a necessary and sufficient condition that $\{W_t\}$ be unitarily equivalent to a direct sum of n copies of $\{S_t\}$ (n may be infinite) is that the von Neumann algebra generated by $\{W_t\}$ be a factor.*

The detailed proof will be published elsewhere, but a sketch may be of interest. The basic fact used (see [2]), is that a nilpotent partial isometry, all of whose powers are partial isometries has a sort of Jordan decomposition. Such an operator is, in fact, the direct sum of operators having a matrix representation:

$$\begin{pmatrix} 0 & 0 & \cdots & & 0 \\ I & 0 & \cdots & & 0 \\ 0 & I & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & & & I & 0 \end{pmatrix}$$