AN INTEGRAL IN TOPOLOGICAL SPACES[1]

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A definition of an integral by majorants and minorants was first given by Perron in [8] for functions of one real variable. Perron's definition was generalized by Bauer in [1] and later on by Mařík in [7], for the case of functions of several real variables whose domains of definition are compact intervals. This generalization, however, was not sufficient, because no adequate substitution theorem holds: even a very simple transformation, e.g., a rotation, of a compact interval is not necessarily a compact interval again. A generalization which admits a relatively suitable substitution theorem was given by the author in [9] and [10]. Although the integral there is defined in a locally compact first countable Hausdorff space, the main emphases are on applications to Euclidean spaces; also only functions with compact domains of definition are integrated.

In this paper we shall define a Perron-like integral in an arbitrary topological space and without any restrictions on the domains of integrable functions. Such generality, of course, causes some changes in basic definitions. Because of omission of the first axiom of countability, the derivate has to be defined by convergence of nets rather than convergence of sequences. Also the possibility of noncompact domains of integration requires a different definition of the majorant.

Throughout P is a topological space and $P^{\sim} = P \cup (\infty)$ is a one-point compactification of P. If $A \subset P^{\sim}$, A^{-} and $A^{\sim}$ denote the closure of A in P and $P^{\sim}$, respectively. For $x \in P^{\sim}$, Γ_x is a local base at x in $P^{\sim}$ (see [6, p. 50]).

Let σ be a nonempty system of subsets of P such that for every A, $B \in \sigma$, $A \cap B \in \sigma$ and $A - B = \bigcup_{i=1}^{n} C_i$ where $C_1, \cdots, C_n$ are disjoint sets from σ.[2] We shall assume that $\Gamma_x \subset \sigma$ for every $x \in P$ and that for each $U \in \Gamma_\infty$ there are disjoint sets $U_{1,\infty}, \ldots, U_{p,\infty}$ from σ such that $U \cap P = \bigcup_{i=1}^{p} U_{i,\infty}$ where the integer $p \geqq 1$ is independent of U.

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[2] Such a system σ is called a prering in [2]. The sets from σ generate the ring over which the integral will be defined. Since our main task is to define a nonabsolutely convergent integral, it should be mentioned here that the system σ must not be too large; e.g., if σ is a σ-ring, then usually only an absolutely convergent integral is obtained. We also note that more general systems than prerings can be used for σ, e.g., a system of all simplexes in a given Euclidean space.