UNIVALENT FUNCTIONS WITH UNIVALENT DERIVATIVES

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1. **Introduction.** Let S denote as usual the family of normalized functions

$$f(z) = z + a_2z^2 + \cdots a_nz^n + \cdots, \tag{1.1}$$

which are holomorphic and univalent in the unit disc D. We consider functions f $(\in S)$ such that most derivatives $f^{(k)}$ are univalent in D. Our conclusion is that f must be entire.

Let E denote those functions f $(\in S)$ which have the property that for each $n \geqq 1$, $f^{(n)}$ is univalent in D. Let

$$\alpha = \sup\{ \mid a_2 \mid ; f \in E\}. \tag{1.2}$$

2. **Main results.** The first theorem shows that if $f \in E$, then f must be an entire function of exponential type. Similar results under somewhat weaker hypotheses are possible. These are contained in the second theorem, the proof of which will appear elsewhere.

THEOREM 1. *If $f \in E$, then for $n \geqq 2$,*

$$\mid a_n \mid \leqq (2\alpha)^{n-1}/n! \tag{2.1}$$

and

$$\pi/2 \leqq \alpha < 1.7208. \tag{2.2}$$

Furthermore, f is an entire function of exponential type such that for every r, we have

$$M(r, f) \leqq \{\exp(2\alpha r) - 1\}/2\alpha. \tag{2.3}$$

PROOF. Since $f^{(n)}$ is univalent in D, $a_{n+1} \neq 0$. Define F_n in D by

$$F_n(z) = \frac{f^{(n)}(z) - n!a_n}{(n+1)!a_{n+1}}.$$

Then $F_n \in E$, and

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