THE IMPOSSIBILITY OF DESUSPENDING COLLAPSES

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It is known that in order to prove the polyhedral Schoenflies conjecture in all dimensions, it is enough to show that, if (B^4, B^3) is a (4, 3) ball pair, then B^4 collapses (polyhedrally) to B^3. Recently, using the solution to the polyhedral Poincaré conjecture in high dimensions, Husch has shown [3] that if (B^7, B^6) is a (7, 6) ball pair, then B^7 collapses to B^6. It is tempting to try to prove that B^4 collapses to B^3 by invoking the following conjecture.

CONJECTURE A. If M is a polyhedral manifold, L a submanifold of M and $S(M)\searrow S(L)$, then $M\searrow L$. ($S(X)$ denotes the suspension of X and "$\searrow$" denotes a polyhedral collapse.)

If Conjecture A were true we could suspend a (4, 3) ball pair three times to obtain a (7, 6) ball pair, use Husch's result, and then apply Conjecture A three times in order to desuspend the collapse.

In this note we present a counterexample to Conjecture A, and discuss other conjectures related to the problem of desuspending collapses.

EXAMPLE 1. Let M^4 be a polyhedral 4-manifold, as described in [4] or [5], with the following properties. (a) M^4 is contractible, (b) $\pi_1(\partial M)\neq 0$, (c) $M^4\times I\cong B^5$. Consider $S(M^4)$ as $M^4\times I$ together with a cone on $M^4\times\{0\}$ and another cone on $M^4\times\{1\}$. Thus if v_0 and v_1 are the vertices of these cones,

$$S(M^4) = (M^4\times I)\cup(v_0*(M^4\times\{0\}))\cup(v_1*(M^4\times\{1\})).$$

Now let B^3 be a 3-ball in ∂M^4. Since $M^4\times I$ is a 5-ball, with $B^3\times I$ as a face, there is an elementary collapse

$$M^4\times I\searrow(M^4\times\{0\})\cup(M^4\times\{1\})\cup[(\partial M^4-\operatorname{int}B^3)\times I].$$

Thus there is a collapse

$$S(M^4)\searrow(v_0*(M^4\times\{0\}))\cup(v_1*(M^4\times\{1\}))\cup((\partial M^4-\operatorname{int}B^3)\times \mathrm{I}).$$

Now, by collapsing conewise $v_i*(M^4\times\{i\})$ to $v_i*((\partial M^4-\operatorname{int}B^3)\times\{i\})$, for $i=0$ and 1, we have $S(M^4)\searrow S(\partial M^4-\operatorname{int}B^3)$. However, since $\pi_1(M^4)=0$ and $\pi_1(\partial M^4-\operatorname{int}B^3)\neq 0$, $M^4\not\searrow\partial M^4-\operatorname{int}B^3$. This provides a counter-example to Conjecture A.

REMARK 1. By taking two copies of the above manifold, M_1 and

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