GENERATING GROUPS OF NILPOTENT VARIETIES

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Problem 14 of Hanna Neumann's book [3] asks for a proof of a conjecture which is contradicted by the following

THEOREM. *If c is an integer greater than 2, then the variety $\mathfrak{N}_c$ of all nilpotent groups of class at most c is generated by its free group $F_{c-1}(\mathfrak{N}_c)$ of rank $c-1$ but not by its free group $F_{c-2}(\mathfrak{N}_c)$ of rank $c-2$.*

In the terms of [3], this means that for $c>2$ one has $d(c)=c-1$ rather than $d(c)=[c/2]+1$ as suggested in Problem 14; correspondingly, [3, 35.35] is false for $c=5$ and 6. (Professor Neumann has confirmed that her proofs were faulty.)

The theorem was suggested by Graham Higman's approach to nilpotent varieties of class c and prime exponent greater than c, via the representation theory of the general linear groups [1]. In particular, he remarked that each critical group in such a variety can be generated by $c-1$ elements (if $c>2$). Since $F_c(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$ (cf. [3, 35.12]) and is residually of prime exponent (cf. Higman [2]), it follows easily that $F_{c-1}(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$. It is not difficult to use Higman's method for confirming the second half of the theorem as well.

In this note we outline a proof which avoids the conceptual complexity of Higman's approach; the price of this is paid for in length. Unless otherwise specified, our notation and terminology follow Hanna Neumann's book [3].

To prove the first half of the theorem, it is sufficient to find a set of homomorphisms from $F_c(\mathfrak{N}_c)$ to $F_{c-1}(\mathfrak{N}_c)$ whose kernels intersect trivially. Hanna Neumann did just this in the proof of [3, 35.35] for $c=4$, and the same idea works generally: if $\{a_1, \cdots, a_c\}$ is a free generating set for $F_c(\mathfrak{N}_c)$ and $\{b_1, \cdots, b_{c-1}\}$ is one for $F_{c-1}(\mathfrak{N}_c)$, then the $2c-1$ homomorphisms $\delta_1, \cdots, \delta_c, \theta_1, \cdots, \theta_{c-1}$ defined by

$$\begin{array}{llllll} a_j\delta_i = b_j & \text{if } j < i, & \text{and} & a_j\theta_i = b_j & \text{if } j \leqq i, \\ a_j\delta_i = 1 & \text{if } j = i, & & a_j\theta_i = b_{j-1} & \text{if } j > i \\ a_j\delta_i = b_{j-1} & \text{if } j > i & & & \end{array}$$

will do. The verification of this makes use of the unique representation of the elements of $F_c(\mathfrak{N}_c)$ in terms of basic commutators in $\{a_1, \cdots, a_c\}$ as defined by Martin Ward [4]. The case of odd c is