THE TOPOLOGICAL DEGREE FOR NONCOMPACT NONLINEAR MAPPINGS IN BANACH SPACES

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Communicated by Felix Browder, December 26, 1967

Let X and Y be Banach spaces, G an open subset of X. If we denote the closure of G by $\mathrm{cl}(G)$, let f be a mapping of $\mathrm{cl}(G)$ into Y.

For $X=Y$ and f a compact mapping, Leray and Schauder [9] gave a definition of topological degree for mappings of the form $I-f$ on the open set G over a point a of X whenever $(I-f)^{-1}(a)$ is a compact subset of G. The Leray-Schauder degree for compact displacements is the most subtle tool of the classical fixed point theory of compact mappings, and if one proposes to carry through an extension of the fixed point and mapping theory to more general mappings which are neither of the form f or $I-f$, with f compact, an important step in such a program is to find a more general framework for the concept of the topological degree. In particular, it would be desirable to bring within such a framework the various classes of nonlinear noncompact operators for which fixed point and mapping results have been obtained in recent years: the asymptotically compact mappings, the nonexpansive and semi-contractive mappings of uniformly convex Banach spaces X into themselves, the monotone and semimonotone mappings of a reflexive Banach space X into its conjugate space X^*, the accretive and semi-accretive mappings of a Banach space X into itself, and their more general analogues (cf. [1], [2], [3], [4]).

It is the object of the present note to present such a framework for a class of mappings which extends that considered by one of the writers in [1]. We give below, by successive stages, the definition of the generalized degree and establish its basic properties.

DEFINITION 1. *Let X be a Banach space, G an open subset of X, T a continuous mapping of* $\mathrm{cl}(G)$ *into X such that $I-T$ is locally compact (i.e. each point of* $\mathrm{cl}(G)$ *has a neighborhood N such that $I-T(N)$ is relatively compact in X) and such that $T^{-1}(a)$ is compact for a given element a of X. Then we define* $\deg(T, G, a)$, *the degree of T on G over a to be* $\deg_{LS}(T, V, a)$, *the Leray-Schauder degree of T over a on any open neighborhood V of $T_-^{1}(a)$ such that $(I-T)(V)$ is relatively compact in X.*

LEMMA 1. *Let T be a mapping satisfying the conditions of Definition* 1. *Then:*

(a) $\deg(T, V, a)$ *is well-defined by Definition* 1 *for any a such that $T^{-1}(a)$ is a compact subset of G.*