ON WEAK MIXING METRIC AUTOMORPHISMS[1]

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Let $(X, \mathfrak{A}, P)$ be a separable probability space and T a metric automorphism of the space onto itself, i.e. T is, except for null sets, a one to one invertible map from X to X such that $T^{-1}(\mathfrak{A}) = \mathfrak{A}$ and $P(T^{-1}A) = P(A)$ for all $A \in \mathfrak{A}$.

There are three standard types of mixing for metric automorphisms, namely ergodic, weak mixing, and strong mixing [2]. It is known [1] that an automorphism is ergodic if and only if for all sets A and B from $\mathfrak{A}$ which have nonzero measure there exists a positive integer n such that $P(T^nA \cap B) > 0$. In this note we show that a similar condition which we call property W is necessary and sufficient for weak mixing.

PROPERTY W. For every two sets A and B of strictly positive measure there exists a subset K of the positive integers with density zero such that for all $k \notin K$, $P(T^kA \cap B) > 0$.

LEMMA. *If a metric automorphism T satisfies property W then it is strongly ergodic, i.e. every nonzero integral power of T is ergodic.*

PROOF. Let m be a given positive integer and A and B two sets of positive measure. Let K denote the set of density zero associated with A and B by property W. Denote by M the set of integers mk where k runs over the positive integers. Since the upper density of M is positive, M is not contained in K and there exists $mk \notin K$. Thus $P(T^{mk}A \cap B) = P((T^m)^kA \cap B) > 0$ and T^m is ergodic. Since T ergodic implies T^{-1} ergodic, T^m is ergodic for all nonzero integers.

THEOREM. *A necessary and sufficient condition that a metric automorphism T be weak mixing is that it have property* W.

PROOF. Suppose first T is weak mixing. Then (see [2]) for A and B given sets of nonzero measure, there exists a subset K' of integers with density zero such that $\lim_{n \notin K'} P(T^nA \cap B) = P(A)P(B) > 0$. Thus for all n not in K' and larger than some integer N, $P(T^nA \cap B) > 0$. Let $K = K' \cup \{k : 0 \leqq k \leqq N,\ k \text{ integer}\}$. The set K has density zero and if $n \notin K$ then $P(T^nA \cap B) > 0$.

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