A NOTE ON COMMUTATIVE ALGEBRA COHOMOLOGY

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This note consists of three parts. First we give an example to show that the commutative algebra cohomology theory described by Gerstenhaber in [4], and in more generality in [3], does not in general vanish in dimension three even when the coefficient module is injective. This implies that the theory cannot be described as the derived functor of the second cohomology group as was done in [1]. In the second part we show that every element of the third cohomology group can be regarded as an obstruction in the sense of Harrison [5]. Finally in §3 we show that the theory may be restricted, without loss of generality, to algebras with unit (and unitary maps).

1. **An example.** Let k be any field and $R=k[x, y]/(x, y)^2$. Then R has a k-basis consisting of $\{1, x, y\}$ with $x^2=y^2=xy=0$. The module $M=\mathrm{Hom}_k(R, k)$, regarded as an R-module by letting $(\alpha f)(\beta)=f(\beta\alpha)$ for $\alpha, \beta\in R$, is well known to be R-injective (see [2, p. 30]). If $\{\epsilon, \xi, \eta\}$ denotes the basis dual to the given one, then $x\xi=y\eta=\epsilon$, $x\epsilon=x\eta=y\epsilon=y\xi=0$.

Let $f : R\otimes R\otimes R\to M$ be the 3-cochain defined on the basis by

$$f(x \otimes x \otimes y) = \xi = -f(y \otimes x \otimes x),$$
$$f(y \otimes y \otimes x) = \eta = -f(x \otimes y \otimes y),$$

and f on any other combination of basis elements should be 0. Then verifying that f is a commutative cocycle is straightforward. Moreover, for any $g: R\otimes R\to M$,

$$\delta g(x \otimes x \otimes y) = xg(x \otimes y) - g(x^2 \otimes y) + g(x \otimes xy) - yg(x \otimes x) \in xM + yM = R\epsilon,$$

which implies that f cannot cobound.

2. **Third cohomology and obstructions.** Let N be a commutative algebra (without unit) and M be its annihilator. Explicitly,

$$M = \{m \in N \mid mN = 0\}.$$

Let N^* be N with a unit adjoined. That is, $N^*=N\times k$ as a k-module

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