JACOBI POLYNOMIAL EXPANSIONS WITH POSITIVE COEFFICIENTS AND IMBEDDINGS OF PROJECTIVE SPACES

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To I. J. Schoenberg on his 65th birthday

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In much of Schoenberg's work there has been a strong interconnection between analytic and geometric reasoning. Here we use a remark he made about imbeddings of metric spaces to prove part of a conjecture about when a Jacobi polynomial $P_n^{(\gamma,\delta)}(x)$ can be expanded in terms of another $P_k^{(\alpha,\beta)}(x)$ with nonnegative coefficients. Also we get from a different special case of this conjecture some nonimbedding theorems for projective spaces.

$P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n, order (α, β), $\alpha, \beta > -1$, is defined by

$$(1)\quad (1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

These polynomials are orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ and what is crucial for us is that $P_n^{(\alpha,\beta)}(1) > 0$. We consider the expansion

$$(2)\qquad P_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} \alpha_k P_k^{(\alpha,\beta)}(x)$$

and ask for what values of $\alpha, \beta, \gamma, \delta$ are all the coefficients α_k, $k=0, 1, \cdots, n$, nonnegative. For $\beta=\delta$ and $\gamma>\alpha$ the α_k were computed by Szegö [8] and were found to be positive. He used this relation to solve the end point Cesàro summability problem for Jacobi series.

For $\alpha=\beta$, $\gamma=\delta$ the α_k were given by Gegenbauer [5] and again they are nonnegative for $\alpha>\gamma$. This has been used by Hua [6] and Askey and Wainger [1]. Actually this result of Gegenbauer is a special case of Szegö's result. For

$$(3)\qquad \frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}$$

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