DENSITIES ON LOCALLY COMPACT ABELIAN GROUPS

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A density on a locally compact Abelian group G is a bounded system of compatible measures on the compact quotients of G. The notion of density arises naturally both in connection with the theory of uniform distribution and with the theory of almost periodic functions. Our main result effectively answers the question "when are all densities on G induced by measures on some compactification of G?" This turns out to be equivalent to the question "when does each density on G extend to a bounded linear functional on the space of semiperiodic functions on G?"

Let G be a locally compact Abelian (LCA) group.

DEFINITION. A density μ on the LCA group G is a system of measures on the compact quotients of G constructed as follows. To each compact quotient Q of G is associated a regular Borel measure μ_Q defined on the Borel sets of Q and satisfying

(1) If E is a Borel set of Q and if Q is a quotient of Q' then $\mu_Q(E) = \mu_{Q'}(\phi^{-1}(E))$, where $\phi\colon Q' \to Q$ is the natural homomorphism.

$$\|\mu\| =_{\text{def}} \sup_Q \{\|\mu_Q\|\} < \infty. \tag{2}$$

An important example of a density is Haar density λ, where λ_Q is simply Haar measure on Q. The densities form a commutative Banach algebra under the obvious operations of addition, scalar multiplication, and convolution $*$ defined by $(\mu * \nu)_Q = \mu_Q * \nu_Q$.

If G is an LCA group and H is a closed subgroup of G such that G/H is compact, then we say that H is a subgroup of compact index in G. We say that a function f on G is periodic if f is constant on the cosets of some subgroup H of compact index, and we then call H a period of f. It is easy to see that a continuous, complex-valued, periodic function on G is necessarily an almost periodic function, and it can be shown easily that an almost periodic function is periodic if and only if the characters which occur in its Fourier expansion generate a discrete subgroup of the dual group $G\hat{}$ of G. In particular, a continuous character χ is periodic if and only if the group generated by χ is a discrete subgroup of $G\hat{}$.

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