ON NONLINEAR PERTURBATIONS OF THE EIGEN-VALUES OF A COMPACT SELF-ADJOINT OPERATOR

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The perturbation theory for an isolated eigenvalue λ_n of a self-adjoint operator L by a linear self-adjoint operator T_θ, depending analytically or continuously on a parameter θ, has been successfully studied since the fundamental results of F. Rellich [10]. In this note we state conditions on nonlinear perturbations $N(u)$ that guarantee the validity of an analogue of Rellich's results. We then show how these results can be applied to qualitative problems in the study of real solutions of nonlinear elliptic partial differential equations and periodic solutions of autonomous ordinary differential equations.

Our study is based on focussing attention on a set of nonlinear invariants for the perturbation problem. These invariants can be considered as a set of critical points of a functional defined on a one parameter family of Hilbert manifolds without boundary ∂A_R.

The critical points are formulated in terms of the Ljusternik-Schnirelmann category as in Palais [9]. We then show that these critical points are stable under the nonlinear perturbation considered.

Previous studies of such nonlinear problems date back to E. Schmidt [11]. Subsequent extensions were made by A. Hammerstein in [5], J. Cronin [4] and R. Bartle [1] among others. Topological methods for such problems were introduced by L. Ljusternik [8], J. Leray and J. Schauder [7], and M. Krasnoselskiĭ [6]. By narrowing the class of perturbations considered, we are able to obtain somewhat sharper results than these previous treatments. This research was partially supported by N.S.F. grant GP 3904 and U. S. Army (Durham) DA-ARO 31-124-G 156. The author is grateful to Professor J. Moser for helpful conversations in connection with this work.

1. Formulation of the problem. Let L be a positive compact self-adjoint operator mapping a real Hilbert space H into itself. Then the operator equation $u=\lambda Lu$ has a countable number of eigenvalues λ_n, each with finite multiplicity. Furthermore the following characterization of λ_n is valid,

$$\lambda_n^{-1} = \max_{[T']_n} \min_{T'} (Lu, u) \tag{1}$$