THE IMMERSION OF MANIFOLDS

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1. M. Hirsch [3] has shown that the immersion problem for manifolds is just a cross section problem for the stable normal bundle. Our object here is to find conditions under which sections of the tangent bundle will imply sections in the normal bundle (and conversely). First we need some notation.

Given an integer t, let $j(t)$ be the maximum integer such that the 2^t-fold Whitney sum of the Hopf bundle over $RP^{j(t)-1}$ is trivial. If ξ is a stable bundle, let $\mathrm{gd}(\xi)$ denote the geometric dimension of ξ.

THEOREM A. *Let M^m be an m-dimensional manifold, $m \leqq 2^t-1$, whose stable tangent bundle τ_0 is trivial over the $(j(t)-1)$-skeleton. If $m-j(t)+1$ is odd or if $H^q(M; Z_p)=0$ for all $p\neq 2$ and $q\neq 0$, m, then* $\mathrm{gd}(\tau_0) \leqq m-j(t)+1$ *implies* $\mathrm{gd}(-\tau_0) \leqq m-j(t)+1$.

To illustrate the strength of Theorem A we offer

THEOREM B. *Let $m=2^t$, then RP^{m-1} immerses in $R^{2m-j(t)+1}$ but not in $R^{2m-j(t)}$.*

The negative result in Theorem B is due to James [4]. Milgram in [8] has obtained linear immersions of RP^m which agree with those of Theorem B only if $m=15$ and 31.

2. **Outline of the proofs.** Let X be a space, then by $X[k]$ we denote the kth-Eilenberg subcomplex of the space X, i.e., $X[k]$ is $(k-1)$-connected and there is a map $f: X[k]\to X$ such that $f_*: \pi_q(X[k]) \cong \pi_q(X)$ for $q\geqq k$. Let BO_n and BO denote, respectively, the classifying spaces of n-plane bundles and stable bundles. The natural map $BO_n\to BO$ induces maps $p: BO_n[k]\to BO[k]$ for all k.

The key step in the proof of Theorem B is

THEOREM C. *For each $m < 2^t$ there exists an H-space E, an H-map $\phi: E\to BO[j(t)]$ and a fiber map $\psi: BO_{n(t,m)}[j(t)]\to E$ such that*

$$\begin{tikzcd} BO_{n(t,m)}[j(t)] \arrow[rr, "p"] \arrow[dr, "\psi"'] & & BO[j(t)] \\ & E \arrow[ur, "\phi"'] & \end{tikzcd}$$

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