WIENER-HOPF OPERATORS AND ABSOLUTELY CONTINUOUS SPECTRA[1]

BY C. R. PUTNAM

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If A is a selfadjoint operator on a Hilbert space $\mathfrak{H}$ with spectral resolution $A = \int \lambda dE_\lambda$, it is known that the set of elements x in $\mathfrak{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ is a subspace, $\mathfrak{H}_a(A)$, reducing A; cf. Halmos [1, p. 104]. In case $\mathfrak{H}_a(A) = \mathfrak{H}$, A is said to be absolutely continuous. The following was proved in Putnam [4]; see also [5] and will be stated as a

LEMMA. *Let T be a bounded operator on a Hilbert space $\mathfrak{H}$ and let*

$$(1) \qquad T^*T - TT^* = C, \qquad C \geqq 0.$$

If $A = T + T^*$, then $\mathfrak{H}_a(A) \supset \mathfrak{M}_T$, where $\mathfrak{M}_T$ is the least subspace of $\mathfrak{H}$ reducing T (that is, invariant under T and T^*) and containing the range of C.

The above will be used to give a short proof of the absolute continuity of certain bounded selfadjoint Wiener-Hopf operators on $L^2(0, \infty)$. For an extensive account of Wiener-Hopf operators on the half-line see Krein [2].

Let $k(t)$, for $-\infty < t < \infty$, satisfy

$$(2) \qquad k \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty) \quad \text{and} \quad k(-t) = \bar{k}(t).$$

Then the operator T on $\mathfrak{H} = L^2(0, \infty)$ defined by

$$(3) \qquad (Tf)(t) = \int_0^t k(s - t)f(s)ds, \qquad 0 \leqq t < \infty,$$

is bounded. (In fact, the hypothesis $k \in L^1(-\infty, \infty)$ alone implies the boundedness of T, even $\|T\| \leqq \int_{-\infty}^{\infty} |k(t)| dt$; cf. Krein [2, pp. 201–202].) The adjoint T^*, which is given by

$$(4) \qquad (T^*f)(t) = \int_t^\infty k(s - t)f(s)ds,$$

and the selfadjoint operator $A = T + T^*$, where

$$(5) \qquad (Af)(t) = \int_0^\infty k(s - t)f(s)ds,$$

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