SOME RESULTS ON LIE p-ALGEBRAS

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Let $\mathfrak{L}$ be a Lie p-algebra ("restricted Lie algebra") over the field $\mathfrak{F}$ of prime characteristic p [3, Chapter V]. Denote by x^p the image of $x\in L$ under the p-power operation, by x^{p^k} the image of x under the kth iterate of $x\to x^p$, with $x^{p^0}=x$. Let $\langle x\rangle$ be the subalgebra of $\mathfrak{L}$ generated by x, i.e., the space of linear combinations of the x^{p^k}, $k=0, 1, 2, \cdots$. Call $x\in\mathfrak{L}$ *separable* if $x\in\langle x^p\rangle$, *nilpotent* if $x^{p^k}=0$ for some k. Then we have proved the following decomposition theorem, which yields a slightly sharpened form of the Jordan-Chevalley decomposition [2, p. 71] for linear transformations in the case of prime chtracteristic.

THEOREM 1. *Let $x\in\mathfrak{L}$, a Lie p-algebra of finite dimension over the perfect field $\mathfrak{F}$. Then there exist elements s, $n\in\langle x\rangle$ with s separable and n nilpotent, such that $x=s+n$. If $y\in\mathfrak{L}$ is separable, $z\in\mathfrak{L}$ nilpotent, $[yz]=0$, and $x=y+z$, then $y=s$ and $z=n$.*

A subalgebra $\mathfrak{T}$ of the Lie p-algebra $\mathfrak{L}$ is called *toral* if $\mathfrak{T}$ is commutative and if every element of $\mathfrak{T}$ is separable. A subalgebra $\mathfrak{N}$ is called *nil* if every element of $\mathfrak{N}$ is nilpotent. For a Lie p-algebra $\mathfrak{L}$ of endomorphisms of a finite-dimensional vector space over an algebraically closed field, to say that $\mathfrak{L}$ is triangulable is to say that $[\mathfrak{L}\mathfrak{L}]$ is nil. In this connection we have the following result.

THEOREM 2. *Let $\mathfrak{L}$ be a Lie p-algebra over the perfect field $\mathfrak{F}$, and suppose that $[\mathfrak{L}\mathfrak{L}]$ is nil. Let $\mathfrak{N}$ be the set of nilpotent elements of $\mathfrak{L}$, and let $\mathfrak{T}$ be any maximal toral subalgebra of $\mathfrak{L}$. Then $\mathfrak{N}$ is an ideal in $\mathfrak{L}$, and $\mathfrak{L}=\mathfrak{T}+\mathfrak{N}$. If, moreover, $\mathfrak{L}$ is nilpotent (as ordinary Lie algebra), then $\mathfrak{T}$ is the set of all separable elements of $\mathfrak{L}$ and $\mathfrak{T}$ is central in $\mathfrak{L}$.*

As to conjugacy of maximal toral subalgebras under these conditions we have shown the following:

THEOREM 3. *Let $\mathfrak{L}$ be a Lie p-algebra over the field $\mathfrak{F}$. Suppose that the set $\mathfrak{N}$ of nilpotent elements is an ideal in $\mathfrak{L}$, and let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be toral subalgebras such that $\mathfrak{T}_i+\mathfrak{N}=\mathfrak{L}$. If $\mathfrak{N}$ is commutative, then there is an automorphism σ of the Lie p-algebra $\mathfrak{L}$ such that $x^\sigma=x$ for all $x\in\mathfrak{N}$, with $y^\sigma-y\in\mathfrak{N}$ for all $y\in\mathfrak{L}$, and with $\mathfrak{T}_1^\sigma=\mathfrak{T}_2$. In general, there is no*

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