ON NORMAL RIEMANNIAN HOMOGENEOUS SPACES OF RANK 1

BY ISAAC CHAVEL[1]

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In this note we shall prove (cf. definitions in [2]) the following

THEOREM. *Let G/H be a simply connected normal Riemannian homogeneous space of rank* 1 *such that every point Q conjugate to $P_0=\pi(H)$ (π is the natural projection) is isotropically conjugate to P_0; then G/H is homeomorphic to a Riemannian symmetric space of rank* 1.

1. Preliminaries. M. Berger [1] has classified the simply connected normal Riemannian homogeneous spaces of rank 1, and with the exception of two, viz, $\mathrm{Sp}(2)/\mathrm{SU}(2)$ and $\mathrm{SU}(5)/(\mathrm{Sp}(2)\times \mathrm{T})$ all are homeomorphic to a Riemannian symmetric space of rank 1. To prove the theorem it therefore suffices to exhibit in each of the spaces $\mathrm{Sp}(2)/\mathrm{SU}(2)$, $\mathrm{SU}(5)/(\mathrm{Sp}(2)\times \mathrm{T})$ a conjugate point of $P_0=\pi(H)$ at which no isotropic Jacobi field vanishes. (A Jacobi field along a geodesic $\sigma(s)$ ($\sigma(0)=P_0$) is *isotropic*, if it is induded by a 1-parameter subgroup of H. A point at which at least one isotropic Jacobi field vanishes is said to be *isotropically conjugate* to P_0.) Furthermore, since the zeros (if they exist) of an isotropic Jacobi field occur only at integral multiples of a fixed real number (Lemma 1 of [2]), it suffices to exhibit in each case a Jacobi field along a geodesic emanating from P_0, vanishing for $s=\alpha$ and not for $s=2\alpha$ ($s=$arc length along the geodesic), such that no Jacobi field with periodic zeros vanishes for $s=\alpha$. In [2] we exhibited the desired Jacobi field for the space $\mathrm{Sp}(2)/\mathrm{SU}(2)$; and we now do the same for the second example.

The equations we will solve will read as:

$$d^2\eta_\alpha/ds^2 + \langle Q_\alpha, [\lambda, Q_\beta]\rangle(d\eta_\beta/ds) + \langle Q_\alpha, [[\lambda, Q_\beta]_{\mathfrak{h}}, \lambda]\rangle\eta_\beta = 0, \tag{1}$$

α, β (repeated indices summed)$=1, \cdots, n=\dim G/H$. In equation (1), s denotes arc length along the geodesic, $\langle\ ,\ \rangle$ the inner product on $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, $\mathfrak{m}=\mathfrak{h}^\perp$, Q_α an orthonormal basis of $\mathfrak{m}$, and $\lambda\in\mathfrak{m}$ the initial unit velocity vector of the geodesic—as usual $\mathfrak{m}$ is identified with the tangent space of $P_0=\pi(H)$ [3]. $[\ ,\ ]$ is the Lie multiplication in $\mathfrak{g}$, and $[\ ,\ ]_{\mathfrak{h}}$ its projection onto $\mathfrak{h}$. We note that the matrices

$$T_{\alpha\beta}(\lambda) = \langle Q_\alpha, [\lambda, Q_\beta]\rangle, \tag{2}$$

$$K_{\alpha\beta}(\lambda) = \langle Q_\alpha, [[\lambda, Q_\beta]_{\mathfrak{h}}, \lambda]\rangle \tag{3}$$

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