A NOTE ON THE MOMENTS OF THE NUMBER OF AXIS-CROSSINGS BY A STOCHASTIC PROCESS

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1. **Introduction.** A general formula, for moments of arbitrary order of the number of upcrossings of a level u by a stationary normal process in unit time, was obtained by Cramér and Leadbetter [1], using a combination of techniques due to Kac [3], and Ylvisaker [6]. Ylvisaker [7] has weakened the conditions of this result slightly by a proof which depends on interesting applications of martingale convergence theory and which may be applied also to nonstationary normal situations. In this note we give a somewhat different direct procedure, under the weakened conditions, for the calculation of these moments. This procedure gives an alternative to that of Ylvisaker [7] for normal processes, without the use of martingale theory, and may be also applied to nonnormal situations in the same way as the discussion in [4] for the first moment.

We shall here give the "counting procedure" used to obtain the number of upcrossings, sketching the derivation, and indicating the extension to nonnormal cases. A detailed proof along these lines (for the stationary normal case) will be given elsewhere (Cramér and Leadbetter [2]).

2. **A general result.** We shall consider a process $x(t)$ possessing, a.s., continuous sample functions and, for a given integer k, absolutely continuous $2k$-dimensional distributions with corresponding densities of the form $f_{t_1 \cdots t_{2k}}(x_1 \cdots x_{2k})$. There will be no loss of generality in considering the number N of upcrossings of the zero level by $x(t)$ in $0 \leqq t \leqq 1$, which is a well-defined random variable (cf. [4]).

For $\boldsymbol{t} = (t_1 \cdots t_k)$ lying in the k-dimensional unit cube, let m_r denote the unique integer such that $m_r/2^n \leqq t_r < (m_r+1)/2^n$. Write $E_n(\boldsymbol{t})$ for the k-dimensional cube whose sides are the intervals $[m_r/2^n, (m_r+1)/2^n)$. For $\epsilon > 0$, let $A_{n\epsilon}$ denote the set of all points $\boldsymbol{t}$ in the unit cube such that for all $\boldsymbol{s} = (s_1 \cdots s_k) \in E_n(\boldsymbol{t})$, we have $|s_i - s_j| > \epsilon$ whenever $i \neq j$, and write $\lambda_{n\epsilon}(\boldsymbol{t})$ for the characteristic function of the set $A_{n\epsilon}$. Finally let the random variable $\chi_{i,n} = 1$ if $x(i/2^n) < 0 < x[(i+1)/2^n]$, $\chi_{i,n} = 0$ otherwise. The following lemma

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