A RELATION BETWEEN MOMENT GENERATING FUNCTIONS AND CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS

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Let X_N for $N=0, \pm 1, \cdots$ be independent random variables with finite first absolute moments; let $A_N=\{a_{N,k}: k=0, \pm 1, \cdots\}$; let $\|A_N\|_\infty=\sup_k |a_{N,k}|$ and $\|A_N\|_p=[\sum_k |a_{N,k}|^p]^{1/p}$ for $1 \leqq p<\infty$; let $S_N=\sum_k a_{N,k}(X_k-EX_k)$; and let p and q be numbers in $[1, \infty]$ satisfying $1/p+1/q=1$.

THEOREM. *Suppose there exist positive constants M, γ, and $1 \leqq p \leqq 2$ such that for $0<x<\infty$ and all values of k*

$$P\{\,|X_k - EX_k| \geqq x\} \leqq \int_x^\infty M \exp(-\gamma t^p)dt. \tag{1}$$

Suppose $\|A_N\|_2$ and $\|A_N\|_q$ are finite for all N. Then

$$T_N = \lim_{\alpha \to -\infty;\, \beta \to \infty} \sum_{k=\alpha}^{\beta} a_{N,k}(X_k - EX_k)$$

exists as an almost sure limit for each N and there exist positive constants C_1 and C_2 such that for every $\epsilon>0$

$$P\{T_N \geqq \epsilon\} \leqq \exp\left[-\min\left\{C_1\left(\frac{\epsilon}{\|A_N\|_2}\right)^2, C_2\left(\frac{\epsilon}{\|A_N\|_q}\right)^p\right\}\right].$$

The constants C_1 and C_2 which are obtained depend only on M, γ, and p. They do not depend in any other way on the distribution of the X_k's and they do not depend on the coefficient sequences A_N.

When $p=1$ the condition (1) is equivalent to the existence of constants $T>0$ and $C>0$ such that $E\exp(tX_k) \leqq \exp(Ct^2)$ for all k and all $|t|<T$; when $1<p \leqq 2$ it is equivalent to the existence of a constant $C>0$ such that $E\exp(tX_k) \leqq \exp[C(t^2+|t|^q)]$ for all k and t.

$$\text{If } p = 1 \text{ and } a_{N,k} = \begin{cases} 1/N & \text{for } k = 1, \cdots, N, \\ 0 & \text{otherwise,} \end{cases}$$

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