REDUCED PRODUCTS, HORN SENTENCES, AND DECISION PROBLEMS

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We shall consider a first order language with equality, and the corresponding structures. *Reduced products* (*powers*) and *Horn sentences* are defined in [4, pp. 199, 211]. An $\forall_n$ *sentence* is a prenex sentence whose prefix consists of a block of universal quantifiers, followed by a block of existential quantifiers, followed by another block of universal quantifiers, and so on n times; $\exists_n$ *sentences* are defined dually. An EC (HC, $\forall_n C$) is the class of all models of an elementary sentence (Horn sentence, $\forall_n$ sentence), and an EC_Δ (HC_Δ, $\forall_n C_\Delta$) is the class of all models of a set of elementary sentences (Horn sentences, $\forall_n$ sentences).

THEOREM 1. *A sentence is preserved by proper reduced products iff it is equivalent to a Horn sentence.*

The "if" part of Theorem 1 is due to Chang [4, Theorem 2.6, p. 215]. The "only if" was conjectured by Chang [8, p. 307] and proved by Keisler [8, result A, p. 307; Corollary 4.3, p. 322] assuming that at least one case of the generalized continuum hypothesis holds, i.e., that $2^\alpha = \alpha^+$ for some infinite cardinal α. Now, I have not found a direct proof of Theorem 1 without using the continuum hypothesis. Instead I have observed that, using Ershov's results in [2], one can eliminate the continuum hypothesis from Theorem 1 by Kreisel's method [11, p. 165] based on Gödel's proof of the consistency of the continuum hypothesis. On the basis of Theorem 1, several related results (Theorems 2, 6, and 8), previously known to follow from the continuum hypothesis, can be proved without assuming that hypothesis and without further recourse to Kreisel's method.

THEOREM 2. *An EC_Δ is closed under proper reduced products iff it is an HC_Δ.*

The "if" in Theorem 2 is due to Chang [4, Corollary 2.7, p. 215]; the "only if" was proved by Keisler [8, Corollary 4.1, p. 322] assum-

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