AN UPPER BOUND FOR RAMSEY NUMBERS

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The finite form of Ramsey's Theorem [2], states that there exists a function $h(n)$ so that if a graph G has at least $h(n)$ points, then either G contains a complete subgraph on n points, or a set of n independent points (a set of points with no edges between any pair). We note that the graphs to be considered have no loops and each pair of points are joined by at most one edge.

Define $f(k, n)$ to be the least integer so that every graph with $f(k, n)$ points contains a complete subgraph on k points or contains a set of n independent points. Erdös and Szekeres [1], proved that

$$f(k, n) \leqq \binom{k+n-2}{k-1}.$$

This upper bound can be improved as we show that

$$f(k, n) \leqq \left(\prod_{i=3}^{k} C_i \right) \frac{n^{k-1}}{(k-1)!} + o(n^{k-1}) \tag{1}$$

where $0 < C_i < 1$ for $i = 3, 4, 5, \cdots$, and in particular

$$f(3, n) \leqq \left(\frac{111 + (33)^{1/2}}{128} \right) \frac{n^2}{2} + o(n^2). \tag{2}$$

DEFINITION 1. A graph G will be called a Ramsey (k, n) graph if it has no complete subgraph on k points and no set of n independent points.

Note that a Ramsey (k, n) graph is not required to have the maximum number, $f(k, n) - 1$, of points.

DEFINITION 2. The complement of a point of a graph G is the subgraph of G obtained by deleting from G the given point, all points joined to this point by an edge and all edges incident to these points.

REMARK 1. The complement of a point in a Ramsey (k, n) graph must be a Ramsey $(k, n-1)$ graph. This is obvious since the point is independent of all points in its complement.

REMARK 2. The set of points joined to a given point in a Ramsey (k, n) graph together with their edges must be a Ramsey $(k-1, n)$ graph.

PROOF OF (1). Let G_n be a Ramsey (k, n) graph on N points. Since