CYCLOTOMIC IDEALS IN GROUP RINGS

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Let G be a group of exponent n and $Z(G)$ the integral group ring of G. We call the ideal of $Z(G)$, which is generated by all elements of the form $1+u+u^2+\cdots+u^{n-1}$ where u runs through all elements of the group G (including 1), the cyclotomic (or Burnside) ideal $\mathfrak{J}(n)$. The motivation for the study of these ideals is the close connection of $\mathfrak{J}(n)$ with the Burnside group $F/R'F^n$ where $G=F/R$ is represented as a factor group of a free group, R' is the commutator subgroup of R, and F^n is the group generated by nth powers of elements of F. Specifically, one considers $RF^n/R'F^n$ as a module over $Z_n(F/RF^n)$ coupled with the observation that for u in $RF^n/R'F^n$ and x in F/RF^n,

$$u^{1+x+\cdots+x^{n-1}} = u(xux^{-1})(x^2ux^{-2}) \cdots (x^{n-1}ux^{-n+1}) = (ux)^n.$$

We hope this announcement helps to initiate a study of the quotient rings $Z(G)/\mathfrak{J}(n)$. We restrict ourselves to the commutative case, where G is a direct product of two cyclic groups of order n. We denote by Σ the augmentation ideal of $Z(G)$; i.e., the ideal generated by $(1-g)$ for all g in G. Throughout, the letter F will always mean a free group of rank 2.

THEOREM 1.
(i) $\Sigma^{p-1}=\mathfrak{J}(p)$, $(\Sigma^{p-2}\not\subset\mathfrak{J}(p))$, *for any prime* p.
(i)i $\Sigma^4\subseteq\mathfrak{J}(4)$, $(\Sigma^3\not\subset\mathfrak{J}(4))$.
(iii) $\Sigma^{12}\subseteq\mathfrak{J}(9)$, $(\Sigma^{11}\not\subset\mathfrak{J}(9))$.

The application to the Burnside groups is the following:

COROLLARY 1.
(i) $F/F''F^p$ *has nilpotency class* $p-1$.
(ii) $F/F''F^4$ *has nilpotency class* 4 *or* 5.
(iii) $F/F''F^9$ *has nilpotency class* 12 *or* 13.

Part (i) of Corollary 1 is due to Meier-Wunderli [1], and Part (i) of Theorem 1 follows easily from his results. In general, our aim is to establish bounds on the nilpotency class of $F/F''F^n$ in the case where $n=p^e$ is a power of a prime, and in the case where n is a composite, to determine where the lower central series becomes stationary. Since we can show the latter is determined once one knows the nilpotency class of $F/F''F^{p_i^e}$ for each prime power p_i^e dividing n, we will restrict