A NONCONSTRUCTIVE PROOF OF GENTZEN'S HAUPTSATZ FOR SECOND ORDER PREDICATE LOGIC

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Takeuti [3] showed that the consistency of analysis (i.e. second order number theory) is finitistically implied by the *Hauptsatz* for second order logic, i.e. by the proposition that every theorem of this system is derivable without cut.[1] We will prove that, conversely, the *Hauptsatz* for this system follows from a certain generalization of the consistency of analysis; namely from:

I. *Every countable set of relations among natural numbers is included in an ω-model.*

An ω-model is a collection of relations among natural numbers which is closed under the second order comprehension axiom. Henkin [1] has shown that a second order formula is derivable with the cut rule if and only if it is valid in all (countable) ω-models.

When the given set of relations consists only of the successor relation, **I** asserts the consistency of analysis.

The formalism for second order predicate logic which we will use is obtained from the system of predicate logic of finite order given in Schütte [2] by dropping all expressions and bound variables of types other than 0 (individuals), 1 (propositions) and $(0, 0, \cdots, 0)$ (relations among individuals). Thus, expressions of type 0 are built up from constants and free variables of type 0 using function constants. The expressions of type $(0, \cdots, 0)$ are constants, free variables and expressions $\lambda x_1^0 \cdots x_n^0 A(x_1^0, \cdots, x_n^0)$, where $A(a_1^0, \cdots, a_n^0)$ is a wff (expression of type 1). The logical symbols other than λ are $\neg$, v and V.

The notation and terminology of [2] will be assumed. In particular, the notions of *strict derivation* and *partial valuation* will be the same as in [2], except that they refer to the second order logic and not the full system of [2], and that we require of a partial valuation that whenever $Vx^\tau A(x^\tau)$ is true (t), then so is $A(a^\tau)$ for some free

[1] Actually, Takeuti asserts this result, not for second order logic, but for G^1LC which contains constants and free variables for third order relations. His proof however is valid for second order logic. On the other hand, the proof of the result of this paper cannot be extended to G^1LC, as far as I know.