

ON A SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ARISING IN MATHEMATICAL ECONOMICS

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Communicated by L. Nirenberg, April 21, 1966

The theory of utility in the social sciences dates back to studies of Daniel Bernoulli, and can be considered as an attempt to extend the ideas of the calculus of variations into the realm of economics. One approach to utility theory consists of deriving a real-valued utility function from a simple system of non-linear partial differential equations, obtainable empirically from economic data. This point of view has been developed for over a century and has been studied recently for equations with continuously differentiable coefficients by Samuelson [6] and Hurwicz and Uzawa [4]. In this paper we consider the case of "kinked" coefficients i.e. Lipschitz continuous but not necessarily everywhere continuously differentiable (a not unrealistic situation in economic behaviour).

Mathematically our results extend a classical theorem of Frobenius (see [2, Chapter VI]). More specifically, we generalize the work of Nikliborc [5], Thomas [8] and Tsuji [9] by using the approach to generalized differentiation as found in Serrin [7]. Our methods also extend to spaces of infinite dimensions in which case we can obtain results useful in the study of the time evolution of economic systems. An (alternative) axiomatic approach to utility theory has been studied in recent years by von Neumann and Morgenstern [10], Herstein and Milnor [3], and others. It is a pleasure to thank Professor L. Hurwicz for suggesting this problem and for invaluable help with its study. This research was partially supported by the grants NSF GP 3904 and AFOSR 883-65.

1. Statement of the problem for finite dimensional commodity spaces. Let $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$ be vectors in \bar{R}^n and R^n respectively with real nonnegative entries. Denote by (p, p_{n+1}) and (x, y) vectors in $R^n \times R^1$ and $\bar{R}^n \times \bar{R}^1$. Then we make following usual definitions of mathematical economics:

(a) A demand function $\tilde{f}(p, m)$ is a mapping of $R^n \times R^1 \rightarrow \bar{R}^n \times \bar{R}^1$ which satisfies the identity:

$$(p, 1) \cdot \tilde{f}(p, m) = m \quad \text{for all } (p, m) \in R^n \times R^1$$

where the dot denotes the usual scalar multiplication in $(n+1)$ dimensions. Economically this identity means: (i) an $(n+1)$ st com-