ON THE STABILITY OF DISCRETIZATIONS[1]

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This paper is concerned with a study of some aspects of the stability question for finite difference methods for the integration of a system of linear ordinary differential equations with constant coefficients. Such a system may be written as follows:

$$du(t)/dt = Mu(t), \qquad t \geqq 0, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where M is a given $n \times n$ complex matrix, $u(t) \in R_n$ for all $t \geqq 0$, and u_0 is a given vector in R_n.

The discrete counterpart of the problem (1), (2) to be considered here is a one-parameter family of discretizations of the form

$$u_k(t + k) = S(k)u_k(t), \qquad t \geqq 0, \tag{3}$$

$$u_k(0) = u_0, \tag{4}$$

where k is a positive real parameter tending to 0, $S(k)$ is an $n \times n$ complex matrix depending only on k, and $u_k(t) \in R_n$ for all $t \geqq 0$, $k > 0$.

DEFINITION 1. The one-parameter family of discretizations (3), (4) is said to be *consistent* with (1), (2) if and only if

$$\| (S(k) - I)/k - M \| \to 0 \quad \text{as} \quad k \to 0 \qquad \text{for any matrix norm.} \tag{5}$$

Various types of stability are introduced in

DEFINITION 2. A one-parameter family of discretizations, $\{B(k)\}$, is said to be

(1) *stable* if and only if for any $T > 0$ there exist positive constants C and k_0 such that $\|S^n(k)\| \leqq C$ for all $0 < k < k_0$ and all positive integers n such that $nk \leqq T$,

(2) *strictly stable* if and only if there exists positive constants k_0 and C such that $\|S^n(k)\| \leqq C$ for all $0 < k < k_0$ and all positive integers n. If k_0 is the largest such constant, then $(0, k_0]$ is called the *interval of strict stability*.

(3) *completely strictly stable* if and only if there exists a positive constant $C(k)$ such that $\|S^n(k)\| \leqq C(k)$ for all $k > 0$ and all positive integers n.

(4) *uniformly strictly stable* if and only if there exists a positive constant C such that $\|S^n(k)\| \leqq C$ for all positive k and nonnegative integers n.

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