MODELS OF SPACE-TIME

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1. **Introduction.** In [1] we exhibited electron spin as a nonrelativistic geometric property of (a model of) Euclidean 3-space. We now extend our model to one of space-time. The connections between 2 and 4 component spinors become lucid, while the Dirac equation and its relativistic "invariance" properties undergo a fundamental simplification and clarification.

2. **Abstract space-time.** We need first an axiomatic foundation strong enough to support both our mathematical considerations and their applications to physics.

DEFINITION. *An $n+1$ dimensional space-time* ($n \geqq 1$) *consists of*

(A) *An $n+1$ dimensional vector space V over the real numbers plus a symmetric bilinear real form $A \cdot B$* (inner product) *such that*:

(1) *There exists a vector A with $A \cdot A < 0$.*

(2) *Any 2-dimensional subspace of V contains a vector A with $A \cdot A > 0$.*

(B) *A set χ of objects $p, q, \cdots$* (points *or* "events") *plus a mapping $(p, q) \rightarrow p-q$ of $\chi \times \chi$ into V such that*:

(1) $(p-q)+(q-r)=p-r$.

(2) *$p-q=0$ implies $p=q$.*

(3) *Given any point q and any vector A there exists a point p with $p-q=A$.*

Any V satisfying (A) yields a model of space-time (*vector space-time*) on setting $\chi = V$. The Minkowski model $V=\chi=R_M^{n+1}$ consists of all $n+1$-tuples of real numbers $x=(x_1, \cdots, x_n, x_{n+1})$ with $x \cdot y = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$. (When $n=3$, $x_4=ct$, where t is time and c is the velocity of light.) Every $n+1$ dimensional vector space-time is isomorphic to R_M^{n+1}, but this result is physically misleading. Eventually we set $n=3$, $\chi=$ the physical space-time continuum, and $V=\mathfrak{S}_4$, the *spin model of* (vector) *space-time* we shall construct.

3. **The models $\mathfrak{S}_3$ and W_4.** In [1] we defined the spin model $\mathfrak{S}_3$ of Euclidean 3-space as the vector space of self-adjoint linear transformations of trace 0 in a 2-dimensional unitary space H_2 (spin space) plus the operations $A \cdot B = (1/2)(AB+BA)$ and $A \times B = (1/2i)$

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