TRIGONOMETRIC SERIES WITH POSITIVE PARTIAL SUMS

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The following problem was proposed by J. E. Littlewood about 15 years ago: Let $S(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ be a trigonometric series having the property that all its partial sums are positive. Is such a series necessarily a Fourier series? The purpose of this note is to show that such is not the case. It is well known that such a series must be a Fourier-Stieltjes series, and, as was shown by H. Helson, even the weaker condition

$$\int |S_n(x)|\, dx < \text{const.}, \qquad \left(S_n(x) = \sum_{-n}^{n} c_j e^{ijx}\right) \tag{1}$$

implies $c_n = o(1)$ (cf. Zygmund [2, p. 286]). It has been shown by Mary Weiss [1] that condition (1) does not imply that $S(x)$ is a Fourier series.

LEMMA 1. *There exists a constant $\alpha > 0$ such that for every $\epsilon > 0$ there exists a real valued trigonometric polynomial $P_\epsilon(x)$, with vanishing constant coefficient, having the properties:*

(i) $|\hat{P}(j)| < \epsilon$,

(ii) *$P_\epsilon(x) > \alpha$ on a set of measure $> \alpha$,*

(iii) *The absolute values of the partial sums of $P_\epsilon(x)$ are less than* 1/2.

PROOF. There exists a constant C such that $|(1/\sqrt{N}) \sum_1^N e^{in \log n} e^{inx}| < C$ (cf. Zygmund [2, p. 199]). Take $N > \epsilon^{-2}$ and $P_\epsilon(x) = \mathrm{Re}((1/2C\sqrt{N}) \sum_1^N e^{in \log n} e^{inx})$. Properties (i) and (iii) are obvious. Property (ii) follows from the fact that

$$\|P_\epsilon\|_{L^2} = \frac{1}{2\sqrt{(2)}C}, \qquad \sup |P_\epsilon(x)| \leqq \frac{1}{2}\cdot$$

We shall also need the following lemma:

LEMMA 2. *Let $f_j(x)$ be real valued trigonometric polynomials satisfying:*

(a) $\hat{f}(0) = 0$,

(b) *$f_j(x) > \epsilon$ on a set of measure $> \alpha$,*

(c) $|f_j(x)| < 1/2$.

Then, if $\lambda_j \to \infty$ fast enough, the product