A NOTE ON FREE SUBSEMIGROUPS WITH TWO GENERATORS[1]

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Let Σ be an alphabet and $F(\Sigma)$ the free semigroup of words over Σ [**3**]. Let $E=\{w_1, w_2\} \subset F(\Sigma)$, where w_1 and w_2 are nonempty words. We give a necessary and sufficient condition on E in order that the subsemigroup generated by E be free with E as its unique irreducible generating set.

DEFINITION. If $w, z \in F(\Sigma)$ are such that $w=z^n$, $n \geqq 0$, then z is a *root* of w. The root of w having minimum word length (maximum n) is the *primitive root* of w and is denoted by $\rho(w)$.

REMARK. It follows from results in [**2**], [**3**], [**4**] that $\rho(w^k)=\rho(w)$ for any $k>0$. Also, if $w_1w_2=w_2w_1$ and w_1, w_2 are both nonempty words, then $\rho(w_1)=\rho(w_2)$. The converse is obviously true; i.e. if $\rho(w_1)=\rho(w_2)$, then $w_1w_2=w_2w_1$.

THEOREM. *Let $E=\{w_1, w_2\}$ be a set of two nonempty words in $F(\Sigma)$. The subsemigroup generated by E is free with E as its unique irreducible generating set if and only if $\rho(w_1)\neq\rho(w_2)$.*

PROOF. Suppose $\rho(w_1)=\rho(w_2)=z$; then by the remark above, $w_1w_2 = w_2w_1$ is a nontrivial relation which implies that the subsemigroup generated by E is not a free semigroup with E as its unique irreducible generating set.

Now suppose that the subsemigroup generated by E is not free with E as its unique irreducible generating set. Then there exists a nontrivial relation of the form

$$w_1^{n_1} w_2^{n_2} \cdots = w_2^{m_1} w_1^{m_2} \cdots, \tag{1}$$

with $n_i>0$, $m_i>0$. Without loss of generality, we may assume that the length of w_2 is not greater than the length of w_1. This together with (1) implies that w_2 is a left-factor of w_1. Hence, we have

$$w_1 = w_2^q t_0, \tag{2}$$

where $q\geqq 1$ is taken to be such that w_2^q is the largest power of w_2 which is a left-factor of w_1. Substituting in (1), we obtain $(w_2^q t_0)w_2 \cdots = w_2^{m_1}(w_2^q t_0) \cdots$,

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