A COMBINATORIAL THEOREM FOR STOCHASTIC PROCESSES

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Let $\{\chi(u),\ 0 \leqq u \leqq t\}$ be a stochastic process where t is a finite positive number. We associate a stochastic process $\{\chi^*(u),\ 0 \leqq u < \infty\}$ with $\{\chi(u),\ 0 \leqq u \leqq t\}$ as follows: $\chi^*(u) = \chi(u)$ for $0 \leqq u \leqq t$ and $\chi^*(t+u) = \chi^*(t) + \chi^*(u)$ for $u > 0$. If the finite dimensional distributions of $\{\chi^*(v+u) - \chi^*(v),\ 0 \leqq u \leqq t\}$ are independent of v for $v \geqq 0$, then the process $\{\chi(u),\ 0 \leqq u \leqq t\}$ is said to have *cyclically interchangeable increments*. In particular, if $\{\chi(u),\ 0 \leqq u \leqq t\}$ has stationary, independent increments, and $P\{\chi(0) = 0\} = 1$, then it belongs to this class.

THEOREM. *If $\{\chi(u),\ 0 \leqq u \leqq t\}$ is a separable stochastic process with cyclically interchangeable increments and if almost all sample functions are nondecreasing step functions which vanish at $u = 0$, then*

$$P\{\chi(u) \leqq u \text{ for } 0 \leqq u \leqq t \mid \chi(t)\} \tag{1}$$
$$= \begin{cases} \left(1 - \dfrac{\chi(t)}{t}\right) & \textit{if } 0 \leqq \chi(t) \leqq t, \\ 0 & \textit{otherwise}, \end{cases}$$

with probability 1.

PROOF. Let $\chi^*(u)$, $0 \leqq u < \infty$, be a nondecreasing step function (nonrandom) for which $\chi^*(0) = 0$ and $\chi^*(t+u) = \chi^*(t) + \chi^*(u)$ if $u > 0$ where t is a fixed positive number. For $u \geqq 0$ define

$$\xi(u) = \begin{cases} 1 & \text{if } \chi^*(v) - \chi^*(u) \leqq v - u \text{ for } v \geqq u, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Obviously $\xi(u+t) = \xi(u)$ for all $u \geqq 0$. Now we shall prove that

$$\int_0^t \xi(u)\,du = \begin{cases} t - \chi^*(t) & \text{if } 0 \leqq \chi^*(t) \leqq t, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The case $\chi^*(t) \geqq t$ is obvious. Thus we suppose that $0 \leqq \chi^*(t) < t$. Define

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