ON THE FACIAL STRUCTURE OF CONVEX POLYTOPES[1]

BY BRANKO GRÜNBAUM

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A finite family $\mathbf{C}$ of convex polytopes in a Euclidean space shall be called a *complex* provided

(i) every face of a member of $\mathbf{C}$ is itself a member of $\mathbf{C}$;

(ii) the intersection of any two members of $\mathbf{C}$ is a face of both.

If P is a d-polytope (i.e., a d-dimensional convex polytope) we shall denote by $\mathbf{B}(P)$ the *boundary complex* of P, i.e., the complex consisting of all faces of P having dimension $d-1$ or less. By $\mathbf{C}(P)$ we shall denote the complex consisting of all the faces of P; thus $\mathbf{C}(P)=\mathbf{B}(P)\cup\{P\}$. For a complex $\mathbf{C}$ we define set$(\mathbf{C})=\bigcup_{C\in\mathbf{C}} C$. For an element C of a complex $\mathbf{C}$ the *closed star* [*anti-star*] of C (in $\mathbf{C}$) is the smallest subcomplex of $\mathbf{C}$ containing all the members of $\mathbf{C}$ which contain C [do not meet C]. The *linked complex* of C in $\mathbf{C}$ is the intersection of the closed star of C with the anti-star of C.

A complex $\mathbf{C}$ is a *refinement* of a complex $\mathbf{K}$ provided there exists a homeomorphism ϕ carrying set$(\mathbf{C})$ onto set$(\mathbf{K})$ such that for every $K\in\mathbf{K}$ there exists a subcomplex $\mathbf{C}_K$ of $\mathbf{C}$ with $\phi^{-1}(K)=\text{set}(\mathbf{C}_K)$.

For example, the complex $\mathbf{K}_1$ consisting of two triangles with a common edge is a refinement of the complex $\mathbf{K}_2$ consisting of one triangle; note, however, that the 1-skeleton of $\mathbf{K}_1$ is *not* a refinement of the 1-skeleton of $\mathbf{K}_2$. Let Δ^d denote the d-simplex. The following result is simple but rather useful:

THEOREM 1. *For every d-polytope P the complex* $\mathbf{C}(P)$ *is a refinement of* $\mathbf{C}(\Delta^d)$.

PROOF. The assertion of the theorem is obviously equivalent to the following statement:

THEOREM 1*. *For every d-polytope P the complex* $\mathbf{B}(P)$ *is a refinement of* $\mathbf{B}(\Delta^d)$.

We shall prove the theorem in the second formulation, using induction on d. The case $d=1$ being trivial, we may assume $d\geqq 2$. Let V be a vertex of P and let H be a $(d-1)$-plane intersecting (in relatively interior points) all the edges of P incident to V. Then $P_0=P\cap H$ is a $(d-1)$-polytope, and, by the inductive assumption, $\mathbf{B}(P_0)$ is a refinement of $\mathbf{B}(\Delta^{d-1})$. Let $\mathbf{S}$ denote the closed star of V in $\mathbf{B}(P)$.

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