ON HOMEOMORPHISMS OF THE PLANE, AND THEIR EMBEDDING IN FLOWS

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Let T be an orientation-preserving homeomorphism of the plane E^2 onto itself. Suppose that T has no fixed points, i.e., that on the sphere $S^2 = E^2 \cup \{\infty\}$, T has no fixed point other than ∞. Then, by the Brouwer Translation Theorem [1], one can construct a Jordan curve $\Gamma \subset S^2$ in such a manner that $\Gamma \cap T\Gamma = \{\infty\}$. If p is a preassigned point in E^2, then Γ can be made so that p lies between Γ and $T\Gamma$.

In the present note we state two new results (Lemma 1 and Theorem 1) about the homeomorphisms treated in Brouwer's theorem. The proofs, which depend to some extent upon Brouwer's techniques, will appear elsewhere. Having presented these results, we will apply them to the problem of embedding homeomorphisms in flows. Indeed, we obtain a natural characterization of those homeomorphisms of the plane which are equivalent to translations.

LEMMA 1. *Let T be an orientation-preserving homeomorphism, without fixed points, of E^2 onto itself. Let $C \subset E^2$ be compact and connected. If $C \cap TC = 0$, then $C \cap T^nC = 0$ for all $n \neq 0$.*

REMARK. This lemma can be used to simplify some of the proofs in Brouwer's original paper.

The second result, like the Translation Theorem, shows how arbitrary homeomorphisms resemble the special homeomorphism $(x, y) \to (x+1, y)$.

THEOREM 1. *Let T be a homeomorphism of E^2 onto itself. Suppose that T preserves the sense of orientation and has no fixed points. Then for any compact set $A \subset E^2$ there is an unbounded connected set $B \subset E^2$ which does not meet $\bigcup_{-\infty}^{+\infty} T^nA$.*

REMARK. One can show by examples that the conclusion of the theorem is false if either of the two assumptions is dropped, or if E^2 is replaced by $E^1, E^3, E^4, \cdots$.

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