A SHARP FORM OF THE VIRIAL THEOREM[1]

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In its classical form the Virial Theorem concerns the behavior of a system S of n mass particles acting under Newtonian attraction in such a fashion that the center of mass 0 remains fixed and the potential energy V satisfies $V > -\infty$ for all positive time t. The latter condition, which is not always stated explicitly, guarantees the analyticity of the coordinates of the particles in the independent variable t; in particular, it excludes collisions [2, pp. 324 ff.]. Let T denote the kinetic energy and h the (constant) total energy $T+V$. Let $\hat{V}$ denote the time average

$$\hat{V} = \lim_{t\to\infty} \frac{1}{t} \int_0^t V(\tau)\, d\tau$$

if the limit exists, with an analogous definition of $\hat{T}$. Clearly each of $\hat{V}$, $\hat{T}$ exists if the other does and $\hat{T}+\hat{V}=h$. The usual theorem states that if S is bounded, in the sense that distances between particles and the velocities of the particles remain bounded, then $\hat{T}$ and $\hat{V}$ exist and $2\hat{T} = -\hat{V}$. An equivalent conclusion is

$$\hat{T} = -h. \tag{1}$$

In this form the theorem is mathematically unsatisfactory because the condition of boundedness is far from necessary. This is already demonstrated by the parabolic case $h=0$ of the two-body problem, $n=2$. In this case distance grows like $t^{2/3}$, so that V behaves like $-t^{-2/3}$ as $t\to\infty$. Consequently, $\hat{V}=0$. Hence $\hat{T}=0$, which is consistent with (1).

We shall replace boundedness by a condition which is both necessary and sufficient. Let $r_{jk}(t)$ denote the distance between particle j and particle k at time t, and let $R(t) = \max_{j,k} r_{jk}(t)$.

THEOREM 1. (1) *is true if and only if*

$$R(t) = o(t), \qquad t \to \infty. \tag{2}$$

Let $2I$ denote the moment of inertia of the system with respect to 0. We begin by showing that (2) is equivalent to

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