JORDAN ALGEBRAS OF DEGREE 1

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For finite-dimensional special Jordan algebras, A. A. Albert proved [1] the following

THEOREM. *Let A be a Jordan algebra over a field Φ of characteristic $\neq 2$. If A has an identity element 1 such that every $a \in A$ is of the form $a = \lambda 1 + z$ for $\lambda \in \Phi$, z nilpotent, then $A = \Phi 1 + Z$ where Z is a nil subalgebra.*

The concepts of inverses and ternary compositions in Jordan algebras were first introduced by N. Jacobson to adapt Albert's argument to the general finite-dimensional case. The following proof shows that the theory of inverses as developed in [2], [3] and subsequently can be used to yield the result directly without any dimensionality restrictions.

PROOF. Let $Z = \{z \mid z \text{ nilpotent}\}$; note that $\lambda 1 + z$ is regular if $\lambda \neq 0$ since it has an inverse in $\Phi[z]$, so $Z = \{z \mid z \text{ singular}\} = \{z \mid U_z \text{ singular}\}$, and hence the fundamental formula $U_{U(x)z} = U_x U_z U_x$ shows $U_x z \in Z$ for all $x \in A$, $z \in Z$. We must show Z is a subalgebra; by commutativity it suffices to show $a, b \in Z \Rightarrow \lambda a, a^2, a+b \in Z$. Clearly $\lambda a \in Z$, and $a^2 \in Z$ since A is power-associative. Suppose $a+b \notin Z$; multiplying by a scalar we may assume $a+b = 1-4z$ for $z \in Z$. But $1-4z$ has a regular square root: if $\lambda_1 = 1$, $\lambda_k = \sum_{1 \leq i \leq k-1} \lambda_i \lambda_{k-i}$ then the λ_k are integers, $w = \sum_{k \geq 1} \lambda_k z^k$ is a well-defined nilpotent element of A, $w - w^2 = z$, so $c = 1 - 2w$ is regular and $c^2 = 1 - 4w + 4w^2 = 1 - 4z$. Thus $a+b = c^2 = U_c 1$, so $a' + b' = U_{c^{-1}} a + U_{c^{-1}} b = U_{c^{-1}} U_c 1 = 1$ for $a', b' \in Z$, which is a contradiction since $b' = 1 - a'$ is regular if a' is nilpotent. Hence we must have $a+b \in Z$.

REFERENCES

1. A. A. Albert, *A theory of power associative commutative algebras*, Trans. Amer. Math. Soc. **69** (1950), 503–527.

2. N. Jacobson, *A theorem on the structure of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 140–147.

3. ———, *A coordinatization theorem for Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1154–1160.

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