THE REAL VALUES OF AN ENTIRE FUNCTION[1]

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The object of this note is to prove the following

THEOREM. *Let $f(z)$ be a nonconstant entire function of order ρ, and let $\Phi(r)$ be the number of points on the circumference $|z| = r$ at which $f(z)$ is real. Then*

$$\limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r} = \rho. \tag{1}$$

We will denote the left-hand side of (1) by κ. Before we proceed with the proof we remark that H. S. Wilf has shown in [3] that $\kappa \geqq \rho$. He then raised the question whether strict inequality could ever hold. The above result settles this problem.

Our proof consists of two parts. In the first, we again prove that $\kappa \geqq \rho$, but by a considerably simpler method than the one given in [3]. The present method has the advantage that it may be used to count not only the real values, but also the values which belong to an arbitrary given unbounded curve. This latter generalization, along with a similar investigation for meromorphic functions, will be published elsewhere [1]. In the second part of the proof we show that $\kappa \leqq \rho$.

PROOF OF THE THEOREM.

Part (i): $\kappa \geqq \rho$. Put $w = f(z)$ and denote by $n(r, w_0)$ the number of w_0 points (multiplicity included) of $f(z)$ in the disc $|z| \leqq r$. A well-known theorem of Borel asserts that for all but at most one finite value w_0,

$$\limsup_{r\to\infty} \frac{\log n(r, w_0)}{\log r} = \rho. \tag{2}$$

Let w_0 be any real value for which (2) holds. Then, an elementary argument implies that there exists an unbounded, increasing sequence $\{r_k\}$ such that

$$\lim_{k\to\infty} \frac{\log n(r_k, w_0)}{\log r_k} = \rho, \tag{3}$$

while $f(z) \neq w_0$ on the circumferences $|z| = r_k$ $(k = 1, 2, \cdots)$.

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