UPPER BOUNDS FOR PERMANENTS OF (0, 1)-MATRICES

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1. **Introduction.** A matrix is said to be a (0, 1)-*matrix* if each of its entries is either 0 or 1. If $A=(a_{ij})$ is an n-square matrix then the *permanent* of A is defined by

$$p(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. Bounds for permanents of general (0, 1)-matrices and for permanents of certain subclasses of (0, 1)-matrices are of combinatorial significance and yet virtually the only known upper bound for $p(A)$ is the obvious one, the product of row sums of A. It has been conjectured that the permanent of an n-square (0, 1)-matrix with exactly k ones, $k<n$, in each row and column must exceed $n!(k/n)^n$ [1, p. 59]. It has been also conjectured by H. J. Ryser that in the class of all mk-square (0, 1)-matrices with exactly k ones in each row and column the maximum permanent is equal to $(k!)^m$, i.e., to the permanent of the direct sum of k-square matrices all of whose entries are 1. In the present note I give a significant upper bound for the permanent of a general (0, 1)-matrix. I also conjecture an upper bound which would allow one to answer Ryser's conjecture in the affirmative.

2. **Results.**

LEMMA. *If $r_1, \cdots, r_c$ are positive integers then*

$$\sum_{j=1}^{c} \frac{2}{r_j} \prod_{t=1}^{c} \frac{r_t}{r_t+1} \leqq 1$$

with equality if and only if $c \leqq 2$ and either r_1 or r_2 is equal to 1.

PROOF. Let E_s denote the sth elementary symmetric function of the numbers $1/r_1, \cdots, 1/r_c$; then

$$(1) \qquad 0 \leqq \prod_{t=1}^{c} (1 - 1/r_t) = 1 - E_1 + E_2 - E_3 + \cdots + (-1)^c E_c$$

with equality if and only if one of the r_t is 1. Therefore

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