UNKNOTTING S^1 IN S^4

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Topologists have for some time suspected that the k-sphere S^k can be topologically knotted in the n-sphere S^n if and only if $k>0$ and $n-k=2$. Strictly speaking, this is not quite correct (because of the existence of wild embeddings), but with the appropriate local flatness condition, the conjecture has been verified by Brown [1; 2] for $n-k=1$, Artin [3] for $n-k=2$, and Stallings [4] for $n-k\geqq 3$, the single undecided case occurring when $k=1$ and $n=4$.

It is the object of this note to show that, on the basis of some recent results of Homma, S^1 can not be knotted in S^4.

1. **The main theorem.** R^n will denote n-dimensional Euclidean space, and we identify R^n with $R^n\times 0\subset R^{n+1}$ so that we may write $R^n\subset R^{n+1}$. The unit sphere in R^{n+1} will be denoted by S^n. S^n can be triangulated as a combinatorial manifold so that, for each $k<n$, S^k appears as a subcomplex.

Let f be an embedding of a k-manifold M^k in an n-manifold M^n with the property that each point of $f(M^k)$ has a neighborhood U in M^n such that the pair $(U, U\cap f(M^k))$ is homeomorphic to the pair (R^n, R^k). Then f is called a *locally flat* embedding and $f(M^k)$ is called a *locally flat* submanifold of M^n.

The main theorem of this paper will be

THEOREM 1.1. *Let f_1 and f_2 be locally flat embeddings of S^1 in S^4. Then there is a homeomorphism h of S^4 onto itself such that*

$$hf_1 = f_2.$$

Furthermore, if p is a point of $S^4-f_1(S^1)-f_2(S^1)$, then h can be chosen so as to restrict to the identity in some neighborhood of p.

Since a general position argument will prove Theorem 1.1 whenever f_1 and f_2 happen to be piecewise linear embeddings, it will be more than sufficient to prove the following theorem, in which $U_\epsilon(f(S^1))$ denotes the set of points in S^4 whose distance from $f(S^1)$ is less than ϵ.

THEOREM 1.2. *Let f be a locally flat embedding of S^1 in S^4. Then for any $\epsilon>0$, there is an ϵ-homeomorphism h of S^4 onto itself such that*

$$h/S^4 - U_\epsilon(f(S^1)) = 1,$$

$$hf\colon S^1 \to S^4 \textit{ is piecewise linear.}$$