PARTIAL RESULTS REGARDING WORD PROBLEMS AND RECURSIVELY ENUMERABLE DEGREES OF UNSOLVABILITY

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Introduction and summary of results. With the settling of the Problem of Post by Friedburg and Mucnik[2] a question that naturally presents itself is whether or not unsolvability results about word problems and related problems can be paralleled for arbitrary recursively enumerable degrees of unsolvability, i.e., for any such degree, D, does there exist a problem of such-and-such a kind having degree D? The present results furnish a partial answer to this general question. Throughout our statement of results, existence is intended in the strong sense of the exhibition of a uniform procedure for constructing.

Corresponding to a well-known unsolvability result of Markov [12][3] and Post [20] we have the following.

RESULT A. *For any recursively enumerable degree of unsolvability, D, there exists a Thue system, $\mathfrak{T}_D$, such that the word problem for $\mathfrak{T}_D$ is of degree D.*

Corresponding to an unsolvability result noted by Marshall Hall [7], we have the following.

RESULT B. *Result A may be strengthened to require that $\mathfrak{T}_D$ be a Thue system on two symbols.*

The following result corresponds more closely to the unsolvability result of WP, §36, than to the unsolvability of the word problem for groups as usually formulated.

RESULT C. *For any recursively enumerable degree of unsolvability, D, there exists a group presentation, $\mathfrak{T}_D$, consisting of a finite number of generators and an infinite but recursive set of defining relations, such that the word problem for $\mathfrak{T}_D$ is of degree D.*

As elsewhere noted,[4] "arbitrary degree" analogues of the Markov-

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[2] E. L. Post, Bull. Amer. Math. Soc. **50** (1944), 314, lines 17–22; R. M. Friedburg, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 236–238; A. A. Mucnik, Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 194–197.

[3] "WP" indicates *The Word Problem*, Ann. of Math. **70** (1959), 207–265 and *numbers in square brackets refer to the bibliography of* WP.

[4] Meeting of the Association for Symbolic Logic, Leeds, August 1962. Result B and related results were presented to this meeting. Result A, to the International Congress of Mathematicians, Stockholm, August 1962. Result C, to the Internationales Kolloquium Über Endliche Gruppen, Oberwolfach, June, 1960. Result C was discovered independently by C. R. Clapham.