COVERS AND PACKINGS IN A FAMILY OF SETS[1]

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1. For a finite set S of elements and a family F of subsets of S, a *cover* C of S from F is a subfamily $C \subset F$ such that $\bigcup(C) = S$. A cover C is called minimum if its cardinality $|C|$ is as small as possible. A *packing* D in F is a subfamily of F whose members are disjoint. It is called maximum if its cardinality $|D|$ is as large as possible. Theorem 1 here is relevant to the task of finding minimum covers. Theorem 2, which follows easily from Theorem 1, is the analogous result on maximum packings. Finally, Theorem 3 extends the foregoing to "α-covers."

Minimum covers are equivalent to solutions of the following integer program: Minimize $\sum x_i$ by a vector $x = (x_1, \cdots, x_n)'$ of zeroes and ones for which $Ax \geqq \bar{1} = (1, \cdots, 1)'$. Here A is the zero-one incidence matrix of members of F (columns) versus members of S (rows). Where $\bar{1}$ is replaced by a vector α of arbitrary positive integers, Fulkerson and Ryser call min $\sum x_i$ the α-width of A. In [3] they find a lower bound for the α-width of zero-one matrices A with given row and column sums.

By analogy with α-width, an *α-cover* of S from F, where α is a vector whose components correspond to the members of S, is a subfamily of F of which at least α_i members contain $y_i \in S$. A β-packing in F is a subfamily of F of which at most β_i members contain $y_i \in S$. Where $\alpha_i + \beta_i$ is the number of members of F which contain $y_i \in S$, the complement in F of an α-cover is a β-packing, and conversely.

The Berge-Norman-Rabin theorem [1], which concerns α-covers where each member of F contains exactly two elements, is based on "alternating paths," invented in 1891 by Peterson and used frequently to prove theorems about linear graphs. Theorem 1 generalizes the N-R instance [5] of the B-N-R theorem, where $\alpha = \bar{1}$, by extending the notion of alternating paths to "alternating trees." Analogously, Theorem 2 includes the Berge theorem [2] on maximum matchings (packings of edges) in a graph. By introducing "self-intersecting trees," Theorem 3 generalizes the B-N-R theorem. Its proof, essentially the same as the proof of Theorem 1, is not given.

Although present knowledge on practical algorithms is quite limited, in theory at least minimum covering and related program-

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