ON REAL JORDAN ALGEBRAS

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Communicated by N. Jacobson, March 6, 1962

Let X be a vector space of the finite dimension n over the field R of the real numbers. For a (scalar or vector valued) function f defined in a neighbourhood of $x \in X$ and differentiable in x, the operator

$$\Delta_x^u f(x) = \frac{d}{d\tau} f(x + \tau u) \Big|_{\tau=0}$$

is defined and linear for every $u \in X$.

We consider triples (Y, ω, c) fulfilling the following conditions:

(i) Y is an open and connected subset of X such that $\lambda > 0$ and $y \in Y$ implies $\lambda y \in Y$.

(ii) $\omega = \omega(y)$ is a continuous real-valued function on the closure $\overline{Y}$ of Y that is homogeneous of degree n, positive, and real analytic in Y, and vanishes on the boundary of Y. Furthermore, the Hessian $\Delta_y^u \Delta_y^v \log \omega(y)$ is nonsingular for $y \in Y$.

Let c be a given point in Y and denote by $\sigma(u, v)$ the Hessian of $\log \omega(y)$ at the point $y = c$. Without restriction we may assume that $\omega(c) = 1$ holds. Since $\sigma(u, v)$ is nonsingular, the adjoint transformation A^* (with respect to σ) is defined for every linear transformation A of X. We form the group Σ' of those linear transformations W of X for which $y \rightarrow Wy$ is a bijective map of Y onto itself and for which $\omega(Wy) = \|W\| \omega(y)$ holds identically for $y \in Y$. Here $\|W\|$ denotes the absolute value of the determinant of W. Let Σ be the subgroup of Σ' consisting of the transformations W in Σ' for which $W^* \in \Sigma'$ holds. The triple (Y, ω, c) is called an *Ω-domain*, if (i), (ii) hold and in addition

(iii) Σ acts transitively on Y.

On the other hand, we consider in X a *Jordan algebra*, i.e., a bilinear and commutative composition $(x, y) \rightarrow xy$ of $X \times X \rightarrow X$ fulfilling

$$x^2(xy) = x(x^2y)$$

for every x, y in X. Such a Jordan algebra, that is, the vector space X together with the composition, shall be denoted by A. For every $x \in X$ the mapping $y \rightarrow xy$ determines a linear transformation $L(x)$ of X such that $xy = L(x)y$. Denote by $\tau(x, y)$ the trace of $L(xy)$. Then $\tau(x, y)$ is a symmetric bilinear form on X. The Jordan algebra A is called *semi-simple* if $\tau(x, y)$ is nonsingular. It is known, that a semi-simple Jordan algebra contains a unit element c. Besides the linear