SOLUTION OF THE DIRICHLET PROBLEM BY INTERPOLATING HARMONIC POLYNOMIALS[1]

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1. Introduction. Let D be a bounded simply connected region of the complex z-plane which is regular for the Dirichlet problem, let C be the boundary of D, and let u be a continuous function on C to the real numbers. Some time ago J. L. Walsh [1, p. 517] suggested that it might be possible to define a sequence of harmonic polynomials by coincidence with the values of u in points so chosen on C that the polynomials converge on D to the solution of the corresponding Dirichlet problem. He showed that a convergent interpolation process of this type is available if D is a circular disk [1; 2]. Recently the author [3] raised the general question anew, and Walsh [4] then extended his earlier result to elliptical disks. For both theoretical and practical reasons it seems worthwhile to develop a general convergence theory, and progress toward this goal is announced here in Theorems 3.2 and 3.3 below.

2. Existence and structure of harmonic interpolation polynomials. A harmonic polynomial of degree n can be written as $h(z)=a_0+\sum_{j=1}^{n}(a_jz^j+\bar{a}_j\bar{z}^j)$, where a_0 is real, $a_n\neq 0$, and the bar over a letter denotes conjugate complex. The real and imaginary parts of any finite complex linear combination of the monomials $1, z, z^2, \cdots, \bar{z}, \bar{z}^2, \cdots$ are harmonic polynomials. Therefore *if such a combination vanishes identically on the boundary of a bounded region, then all of the coefficients must be zero.* If $p_1(z), p_2(z), \cdots, p_n(z)$ are any complex polynomials in z of respective degrees $1, 2, \cdots, n$, then the expression $b_0+\sum_{j=1}^{n}[b_jp_j(z)+\bar{b}_j(p_j(z))^-]$ is also a harmonic polynomial of degree n, where now $(p_j(z))^-$ denotes the complex conjugate of $p_j(z)$.

Given points $z_1, z_2, \cdots, z_{2n+1}$ and real or complex numbers $u_1, u_2, \cdots, u_{2n+1}$, a necessary and sufficient condition that there exists a unique set of $2n+1$ numbers $c_0, c_1, \cdots, c_n, d_1, \cdots, d_n$ satisfying the system of linear algebraic equations

$$P(z_h) = c_0 + \sum_{j=1}^{n}[c_jp_j(z_h) + d_j(p_j(z_h))^-] = u_h, \qquad h = 1, 2, \cdots, 2n+1,$$

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