MODEL THEORIES WITH TRUTH VALUES IN A UNIFORM SPACE

BY C. C. CHANG AND H. JEROME KEISLER

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In recent years the ultraproduct construction has been applied, e.g. in [4] and [2], to obtain a series of results in the theory of models for the ordinary two-valued first-order predicate logic. Most of the results in [4] and [2] have been generalized in [1] to predicate logic with truth values in the closed real unit interval. In this note we shall see that many of the methods and results of [4] and [2] and [1] can actually be extended to a very wide class of many-valued predicate logics, with truth values in any reasonably well-behaved compact Hausdorff uniform space.

We shall give a detailed statement of the definitions and two representative theorems. A complete account of the theory, including a number of generalizations of theorems from [2] and [1], as well as proofs, will appear in a future publication.

Let L be a formal system with the following symbols: a denumerable set V of individual variables, a set P of finitary predicates, a set C of finitary sentential connectives, a set Q of quantifier symbols, and distinguished symbols $e \in P$, $\& \in C$, $\exists \in Q$, where e and $\&$ are binary. Let the set F of formulas be the least set H such that

(i) $\{p(v_1, \cdots, v_n) \mid p \in P,\ p$ is n-ary, $v_1, \cdots, v_n \in V\} \subseteq H$;

(ii) $\{c(\phi_1, \cdots, \phi_k) \mid c \in C,\ c$ is k-ary, $\phi_1, \cdots, \phi_k \in H\} \subseteq H$;

(iii) $\{qv(\phi) \mid q \in Q,\ v \in V,\ \phi \in H\} \subseteq H$.

Free variables are defined as usual. ϕ is a sentence if $\phi \in F$ and ϕ has no free variables.

Given sets X, Y, and Z, $S(X)$ shall denote the set of all subsets of X and f: $Y \to Z$ shall mean f is a function on Y into Z.

If X is a uniform space with uniformity $\mathfrak{U}$ (see [3]), a set function $g: S(X) \to X$ is *uniformly continuous* if for each $U \in \mathfrak{U}$, there exists $U' \in \mathfrak{U}$ such that whenever $Y \subseteq X \cap U'[Z]$ and $Z \subseteq X \cap U'[Y]$, then $(g(Y), g(Z)) \in U$. $\mathfrak{X} = (X, f, t, \hat{c}, \hat{q})_{c \in C, q \in Q}$ is a *model theory* if

(i) X is a compact Hausdorff uniform space;

(ii) $f, t \in X$ and $f \neq t$;

(iii) for each k-ary $c \in C$, $\hat{c}$: $X^k \to X$ and $\hat{c}$ is continuous;

(iv) for each $q \in Q$, $\hat{q}$: $S(X) \to X$ and $\hat{q}$ is uniformly continuous.

$\mathfrak{A} = (A, p_{\mathfrak{A}})_{p \in P}$ is a *structure over* X if

(i) $A \neq 0$;

(ii) for each n-ary $p \in P$, $p_{\mathfrak{A}}$: $A^n \to X$;

(iii) for $a, b \in A$, $e_{\mathfrak{A}}(a, b) = t$ if $a = b$, and $e_{\mathfrak{A}}(a, b) = f$ if $a \neq b$.