ON THE EXISTENCE OF INVARIANT MEASURES

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Under the same title the outline of a rather involved proof of the existence of a finitely additive measure invariant under a transformation was recently published in this Bulletin [4]. A more general theorem was announced by the author several years ago in [2]. In this note first the original simple proof (unpublished) will be applied to yield a further generalization of the latter result (Theorem 1). Then its thesis will be strengthened under some of the assumptions made in the above publications (Theorem 2). Finally, some fundamental properties of the invariant set function will be established (Theorem 3).

Let S be a class of subsets of a set X. For each element g of a semigroup G, let $\bar{g}$ be an additive mapping of S into itself such that

$$\bar{h}\bar{g}A = \bar{g}(\bar{h}A) \qquad \text{for all } g, h \in G, A \in S.$$

If m is any set function on S, then call a set function μ defined on a class of sets $\Sigma \supset S$ a *G-perfection of m on* Σ if it has the following properties: (i) it is finitely additive; (ii) it is *G-invariant* i.e. $\mu(\bar{g}A) = \mu(A)$ for all $g \in G$ and $A \in S$; (iii) all its values are between the extreme bounds of those of m; (iv) $\mu(A)$ is between the extreme bounds of the set of all numbers $m(\bar{g}A)$ with $g \in G$ for every $A \in S$.

THEOREM 1. *If the semigroup G is abelian (or, more generally, left measurable*[2]*), then every finitely additive and bounded set function m on S has a G-perfection on S.*

PROOF. It was shown in [3] that every abelian semigroup G is *left measurable* i.e. that there is a linear functional K on the set B of all bounded real functions on G such that (a) $f \in B$, $f_h(g) = f(hg)$ for all $g \in G$ implies $K(f_h) = K(f)$ for all $h \in G$, (b) $K(1) = 1$, (c) $K(f) \geqq 0$ if $f(g) \geqq 0$ for all g in G. Theorem 1 is proved by straightforward verification that the set function μ defined by the formula

$$\mu(A) = K[m(\bar{g}A)], \qquad A \in S,$$

is a G-perfection of m on S.

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[2] See definition in the proof.