A SIMPLE TRIANGULATION METHOD FOR SMOOTH MANIFOLDS[1]

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We first triangulate a compact closed m-manifold M^m of differentiability class C^r $(r>1)$ in a euclidean space $E^\nu = E^{m+n}$. The method is simpler than earlier methods (see References) and is applicable to a wider class of spaces (see (F) and (G) below).

(A) For a given $\eta>0$, let $(a_1, \cdots, a_\mu)$ be a set of distinct points on M^m such that each point of M^m is at distance $<\eta$ from at least one point a_i.

Let d be the euclidean distance function in E^ν. For each $k\in(1, \cdots, \mu)$, let

$$\bar{c}_k^\nu = \{x \in E^\nu \mid d(a_k, x) \leqq d(a_i, x), \quad i = 1, \cdots, \mu\}, \tag{1}$$

$$\bar{\gamma}_k^m = M^m \cap \bar{c}_k^\nu = \{x \in M^m \mid d(a_k, x) \leqq d(a_i, x), \quad i = 1, \cdots, \mu\}. \tag{2}$$

THEOREM. *For each $p\in M^m$, let $\bar{\gamma}(p)$ be the intersection of all the sets $\bar{\gamma}_k^m$ containing p. If η is small enough, $\{\bar{\gamma}\} = \{\bar{\gamma}(p) \mid p\in M^m\}$ is a subdivision of M^m into the closed cells of a complex.*

PROOF. Note first that if $i\neq k$, $d(a_k, x) = d(a_i, x)$ defines the normal bisecting $(\nu-1)$-plane $L_{ki}^{\nu-1}$ of the segment $a_k a_i$, and $d(a_k, x) < d(a_i, x)$ defines the half-space $H_{ki}^{\nu-1}$ of E^ν bounded by $L_{ki}^{\nu-1}$ and containing a_k. Thus $\bar{c}_k^\nu$ is the closure of the open convex polyhedral ν-cell

$$c_k^\nu = \bigcap_{i\neq k} H_{ki}^\nu = \{x \in E^\nu \mid d(a_k, x) < d(a_i, x), i \neq k\}, \tag{3}$$

which may be of infinite diameter.

(B) *The set $\bar{\gamma}_k^m = \bar{c}_k^\nu \cap M^m$ is on the interior $B^\nu(a_k, \eta)$ of the sphere $S^{\nu-1}(a_k, \eta)$ of radius η about a_k.*

For, by (A), each point of $M^m - \overline{B}^\nu(a_k, \eta)$ is closer to some $a_i \neq a_k$ than to a_k, and c_k^ν is the set of all points which are closer to a_k than to any $a_i\neq a_k$.

The fact that M^m is compact and of class C^2 implies that there exists a number $\rho>0$ so small that no $(\nu-1)$-sphere of radius ρ tangent to M^m encloses a point of M^m. The cell c_k^ν therefore contains all points at distances $\leqq\rho$ from a_k on the normal n-plane $N^n(a_k)$ to M^m at a_k, since each such point is closer to a_k than to any $a_i\neq a_k$.

(C) Hence, if $L_{ki}^{\nu-1}$ (defined above) intersects $\bar{\gamma}_k^m$, then $L_{ki}^{\nu-1}\cap N^n(a_k)$ is either vacuous or at distance $>\rho$ from a_k.

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