INEQUALITIES FOR FORMALLY POSITIVE INTEGRO-DIFFERENTIAL FORMS

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In 1954 N. Aronszajn [1] proved an inequality for formally positive integro-differential forms which has been found very interesting in itself and which has had a strong influence on subsequent progress in elliptic partial differential equations. We propose to extend this inequality in respect to the class of possible domains of integration and in respect to the kind of norms involved.

Let $\{P_j\}$ be a finite set of differential operators of order m with continuous coefficients on the closure $\bar{G}$ of a domain $G \subset R^n$. Suppose that the characteristic polynomials[2] $p_j(x, \xi)$ have no common real zero $\neq 0$ for $x \in \bar{G}$ and no common complex zero $\neq 0$ for $x \in \bar{G} - G$. Then an inequality of the form

$$(1) \qquad \sum_j \int_G |P_j u|^p dx + \int_G |u|^p dx \geq c \int_G |D^m u|^p dx, \qquad p > 1,$$

holds for all functions u of class C^m on $\bar{G}$ and all derivatives $D^m u$ of order m. The inequality is valid for a large class of bounded domains G—finite sums of those with boundary of Lipschitz graph type [**2**; **4**]—including, for example, all with smooth boundary, all convex domains, and all finite sums of such.[3] With minor modifications it is also valid for quite a large class of unbounded domains.

The full details of the statement of the theorem, as well as the proof, will be given in another paper. Here we prefer to show the proof in a case which, though special, still contains the idea. We will suppose:

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[2] The p_j are the polynomials corresponding to the leading parts of the P_j. Thus, they are homogeneous polynomials (in ξ) of degree m. They all have the trivial zero $\xi = 0$.

[3] In Aronszajn's theorem $p = 2$ and there is the (rather inconvenient) restriction that G have a boundary of class C^1. In return, however, the condition on complex zeros is weakened. It is only required that there be none with imaginary part orthogonal to the boundary of G at x. Aronszajn's conditions are necessary as well as sufficient. It is not so clear how they should be re-formulated in the present case of (possibly) irregular boundaries. It should be noted, however, that when G is bounded with boundary of class C^1 and the leading parts of the P_j have constant coefficients our conditions are equivalent to his.