THE MINIMUM MODULUS OF INTEGRAL FUNCTIONS OF SMALL ORDER[1]

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Let $f(z)$ be an integral function and

$$M(r) = \max_{|z|=r} |f(z)|, \qquad m(r) = \min_{|z|=r} |f(z)|.$$

If $f(z)$ has order 0, then the $\cos \pi\rho$-theorem [1, p. 40] implies that, for any $\epsilon > 0$, there is a sequence of $r \to \infty$ on which

$$\log m(r) > (1 - \epsilon) \log M(r). \tag{1}$$

If $\log M(r) = O(\log^2 r)$ $(r \to \infty)$ then Hayman [2] proved that (1) holds outside a set of finite logarithmic measure. If $f(z)$ has order at most ρ and C is any constant, then Kjellberg [3, p. 20] showed that

$$\text{lower log-dens } E\{m(r) > C\} \geqq 1 - 2\rho,$$

and that $1 - 2\rho$ is best-possible. Surveys of the main results of this kind are given in [1; 3].

In this field I have proved a number of results, of which I state the following. For a given function $\psi(r)$ $(r > 0)$ I use the notation

$$\psi_1(r) = d\psi(r)/d \log r, \qquad \psi_2(r) = d^2\psi(r)/d \log^2 r,$$

when these derivatives exist.

THEOREM 1. *Suppose that*

$$\log M(r) \leqq \{1 + o(1)\}\psi(r) \qquad (r \to \infty),$$

that $\psi_1(r) = o\{\psi(r)\}$ $(r \to \infty)$ *and that for* $r \geqq r_0$, $1/\psi_2(r)$ *is positive, monotonic decreasing, and a convex function of* $\log r$. *Then if* $0 < \delta < 1$ *and* $\epsilon > 0$ *we have*

$$\text{lower log-dens } E\left\{\log \frac{m(r)}{M(r)} > -(1 + \epsilon)\delta^{-1}(2 - \delta) \frac{1}{2} \pi^2 \psi_2(r)\right\} \geqq 1 - \delta.$$

We call a function $L(x)$ continuous and positive for $x \geqq 0$, slowly oscillating, if for each fixed $\lambda > 0$,

[1] This, in the main, is an abstract of a thesis submitted for the degree of Ph.D. in the University of London, under the supervision of Professor W. K. Hayman.